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RADIATION OF A COAXIAL LINE INTO A HALF-SPACE

by

John S. Asvestas

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14. ABSTRACT

We present a new approach to the problem of radiation by a coaxial line into a half-space. We obtain boundary-integral equations for the current densities over the walls of the coax and its opening to the half-space. Using the modes of the coax for basis and testing functions, we convert the integral equations to an infinite system of linear equations, the unknowns being the coefficients of the current density expansions. We demonstrate how, by solving a small subsystem, we can obtain the fields everywhere in space.

15. SUBJECT TERMS

Coaxial-line radiation, boundary-integral equations (BIEs), method of moments, modal basis functions, modal testing functions.

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SUMMARY

In 2006, we published a paper that is based on the present report[1]. Because of a restriction on the number of printed pages, we had to omit a good part of the analysis. The omissions were so extensive that even the author could not follow the mathematical trend when looking at the paper a few years later. For this reason, we went back to the original notes and compiled this report that gives a detailed account of the subject matter. In addition to aiding in understanding how the results in the paper were obtained, this report can also guide the reader in extending the results to other geometries, such as a circular or rectangular waveguide.

The problem we deal with here is the classical problem of radiation of a semi-infinite coaxial line into a half-space. With respect to a rectangular coordinate system *xyz*, the coaxial line is along the *z*-axis and originates at minus infinity. The inner conductor terminates at $z = 0$, while the outer one opens up into an infinite plane perpendicular to the coaxial line. Details on the history and bibliography of this problem are given in the aforementioned paper and will not be repeated here. Our work on this problem differs from that of other works in that it is an exact formulation of the problem in terms of boundary integral equations (BIEs). The idea for this approach originated in our work on scattering by an indentation on a ground plane[2]. Its main characteristic is that the domain of the integral equations does not involve the infinite plane.

In this report, we not only provide the missing analysis for the geometry we just described but we also present the problem in a more general setting. In Part 1, we derive BIEs for a monopole over a perfectly conducting plane, driven by a coaxial line. The monopole may or may not be a continuation of the center conductor of the coaxial line and can be quite general in shape. The integral equations extend over the monopole, the opening to free space of the coaxial line and over either the semi-infinite walls of the coaxial line or over a finite part of them and the inter-wall spacing at the end of the finite part.

In Part 2, we employ the equations we found in Part 1 to the problem we presented in the paper. We take advantage of the circular symmetry of the problem to reduce the vector integral equations to three scalar equations. By expressing the unknown current densities in terms of the natural modes of the coaxial line, we show that the problem can be reduced to solving a single, scalar integral equation. We also derive expressions for the far fields in the half-space in which the coaxial line radiates. In Part 3, we take advantage of the orthogonality properties of the modal functions to convert the scalar integral equation into an infinite system of linear algebraic equations. We also convert the coefficients of the system from double to single integrals and proceed to show how to compute them. In Part 4, we provide detailed information on how we compute the system of equations and consider four different coaxial lines for which we compute a number of quantities of interest. In the last part, Conclusions, we summarize the work and offer suggestions for further work.

[1] J. S. Asvestas, “Radiation of a coaxial line into a half-space”, *IEEE Trans. Antennas Propagat.*, Vol. 54, No. 6, pp. 1624-1631, (2006).
[2] J. S. Asvestas and R.E. Kleinman, “Electromagnetic Scattering by Indented Screens,” *IEEE Trans. Antennas Propagat.*, Vol. 42, pp. 22-30, (1994).

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PART 1
INTEGRAL EQUATIONS FOR A MONOPOLE OVER A GROUND PLANE

ABSTRACT

In Part 1, we present a new approach to the problem of monopole radiation over an infinite, perfectly conducting plane. The monopole is fed by an air-filled, infinite, coaxial line. The approach we use is mathematically rigorous and physically exact. It leads to a system of boundary integral equations (BIEs) that can be solved numerically using well established methods.

Part 1 has three chapters. In the first chapter, we derive BIEs that extend over the entire length of the walls of the infinite coaxial line. The monopole is not physically connected to the center conductor of the coaxial line. The BIEs involve the electric surface-current density as the unknown.

In the second chapter, the geometry remains the same as in the first part but the BIEs extend over the part of the walls of the coaxial line that border its connection to the infinite plane. This is accomplished by introducing an additional unknown, namely, the magnetic surface-current density. We show that the results of the first part can be obtained from the results of the second by a series of transformations and we also derive formulas for the far field.

In the third chapter, we make the monopole a natural extension of the center conductor of the coaxial line. We then use the results of the first two parts to obtain BIEs for this, more realistic, case.

We point out that the shape of the monopole is quite arbitrary but that the BIEs are valid only when all materials are perfectly conducting.

CHAPTER 1
EQUATIONS EXTENDING OVER THE ENTIRE WALLS OF THE LINE

1. INTRODUCTION

In this chapter, we formulate the problem of radiation by a monopole over a ground plane in terms of BIEs. The ground plane is infinite and the monopole is fed by an infinite coaxial line. This formulation is the foundation for the rest of the report and contains general comments that will not be found elsewhere in this report.

We derive BIEs for the geometry of Figure 1.1. All surfaces are perfectly conducting. The inner and outer radii of the coaxial line are *a* and *b*, respectively. The line supports only a TEM wave which means that (references 1 and 2)

$$k\frac{a+b}{2}<1 \tag{1.1}$$

where k is the wavenumber of the time-harmonic ($e^{+i\omega t}$) electromagnetic wave in the line.

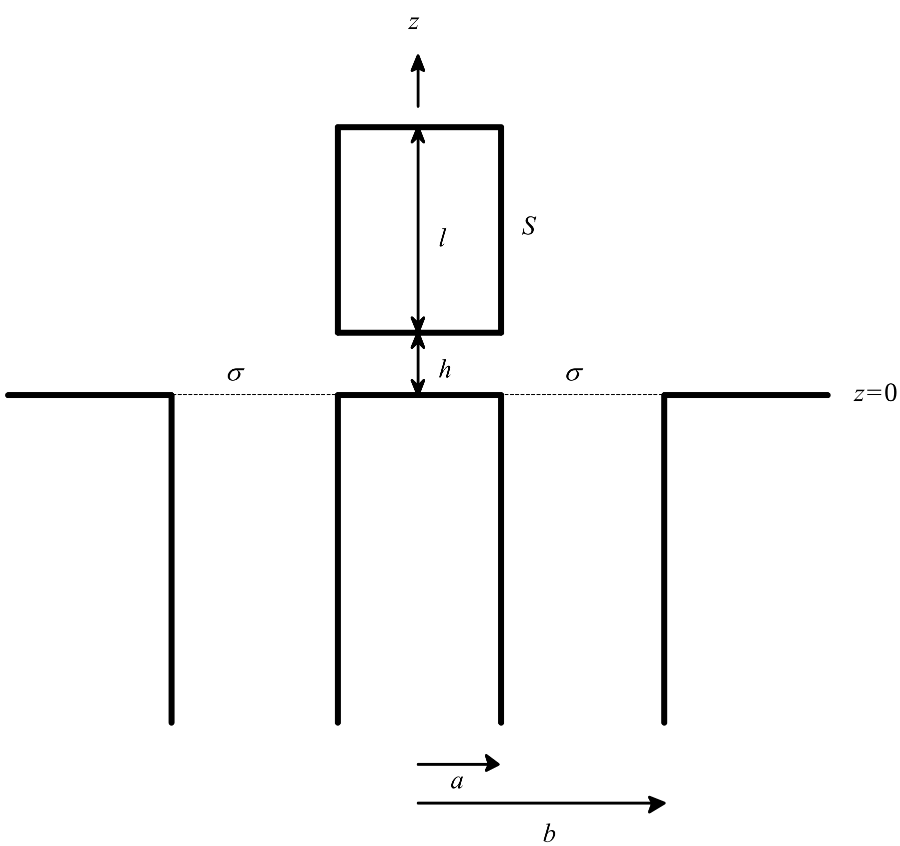

Figure 1.1. A cylindrical monopole over a ground plane, fed by a coaxial line.

The excitation of the line occurs at $z=-\infty$ and results in a TEM wave with fields

$$\mathbf{E}^g(\mathbf{r}) = \frac{V\,\mathrm{e}^{-ikz}}{\ln\left(\dfrac{b}{a}\right)\rho}\,\hat{\rho}\,, \quad \mathbf{H}^g(\mathbf{r}) = \frac{YV\,\mathrm{e}^{-ikz}}{\ln\left(\dfrac{b}{a}\right)\rho}\,\hat{\varphi}\,. \tag{1.2}$$

where V is the voltage of the inner conductor with respect to the outer, and Y is the free-space impedance. We have also employed cylindrical coordinates (ρ, φ, z). Since the line is not infinite, we also have an induced wave in the line with fields $\{\mathbf{E}^i, \mathbf{H}^i\}$. The total fields $\{\mathbf{E}^t, \mathbf{H}^t\}$ in the coaxial line are the sum of the generator and induced fields

$$\mathbf{E}^t = \mathbf{E}^g + \mathbf{E}^i\,, \quad \mathbf{H}^t = \mathbf{H}^g + \mathbf{H}^i\,. \tag{1.3}$$

In the upper-half space, we have radiated fields $\{\mathbf{E}^r, \mathbf{H}^r\}$. We proceed to determine these fields by first deriving integral equations and then solving them numerically.

2. INTEGRAL REPRESENTATIONS IN UPPER HALF-SPACE

In this section, we derive integral representations of the electromagnetic fields in the upper-half space. From these representations, we will eventually obtain integral equations.

We apply Green's second identity in the upper half-space of Figure 1.1, excluding the interior of the monopole. Thus, the surface integrals will extend over the surface S and the infinite plane $z=0$. There is also a surface integral over a hemisphere with center at the origin and whose radius tends to infinity. We omit this integral since the radiating fields satisfy the Silver-Müller condition and, hence, the integral will tend to zero as the radius of the hemisphere tends to infinity. From Green's second identity (reference 3, p. 509), we have

$$\begin{aligned}&\int_{D^+}\left\{\mathbf{E}^r(\mathbf{r})\cdot\left[\nabla\times\nabla\times\Gamma_1(\mathbf{r},\mathbf{r}')\right]-\left[\nabla\times\nabla\times\mathbf{E}^r(\mathbf{r})\right]\cdot\Gamma_1(\mathbf{r},\mathbf{r}')\right\}dV\\&=\int_{(z=0)\cup S}\left\{\left[\hat{n}\times\mathbf{E}^r(\mathbf{r})\right]\cdot\nabla\times\Gamma_1(\mathbf{r},\mathbf{r}')-\nabla\times\mathbf{E}^r(\mathbf{r})\cdot\left[\hat{n}\times\Gamma_1(\mathbf{r},\mathbf{r}')\right]\right\}dS\end{aligned}\tag{2.1}$$

where D^+ is the whole upper-half space except for the region occupied by the monopole, and $\hat{n}$ is the interior unit normal. The normal is equal to $\hat{z}$ at $z=0$ and points to the exterior of the monopole on S. The dyadic Γ_1 has the form

$$\boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')=-ik\nabla\times\left[g(\mathbf{r},\mathbf{r}')\mathbf{I}+g(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i\right]\tag{2.2}$$

where g is the free-space scalar Green's function

$$g(\mathbf{r},\mathbf{r}')=-\frac{\mathrm{e}^{-ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\tag{2.3}$$

and

$$\mathbf{I}=\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}\,,\quad \mathbf{I}_i=\hat{x}\hat{x}+\hat{y}\hat{y}-\hat{z}\hat{z}\tag{2.4}$$

are the identity dyadic and its image about the xy-plane. Moreover, for any vector $\mathbf{a}$, we define its image $\mathbf{a}_i$ about the xy-plane by

$$\mathbf{a}=\left(a_x,a_y,a_z\right)\Leftrightarrow\mathbf{a}_i=\left(a_x,a_y,-a_z\right).\tag{2.5}$$

The dyadic $\boldsymbol{\Gamma}_1$ satisfies the differential equation

$$\nabla\times\nabla\times\boldsymbol{\Gamma}_1\left(\mathbf{r},\mathbf{r}'\right)-k^2\boldsymbol{\Gamma}_1\left(\mathbf{r},\mathbf{r}'\right)=ik\nabla\times\left[\delta(\mathbf{r},\mathbf{r}')\mathbf{I}+\delta(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i\right]\tag{2.6}$$

and the boundary condition

$$\hat{z} \times \boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}') = \mathbf{0} \;\text{ at } z = 0\,. \tag{2.7}$$

The electric field satisfies the reduced wave equation

$$\nabla \times \nabla \times \mathbf{E}^r(\mathbf{r}) - k^2\mathbf{E}^r(\mathbf{r}) = \mathbf{0} \tag{2.8}$$

and the boundary conditions

$$\hat{n} \times \mathbf{E}^r(\mathbf{r}) = \mathbf{0}\,, \quad \mathbf{r} \in S\,; \quad \hat{z} \times \mathbf{E}^r(\mathbf{r}) = \mathbf{0}\,, \quad \mathbf{r} \in \{(x, y, z):\; z = 0, (x,y) \notin \sigma\} \tag{2.9}$$

Substituting (2.6) and (2.8) in I_{D^+}, the left-hand side of (2.1), we find

$$\begin{aligned} I_{D^+} &= ik\int_{D^+} \left\{\mathbf{E}^r(\mathbf{r}) \cdot \nabla \times \left[\delta(\mathbf{r},\mathbf{r}')\mathbf{I} + \delta(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i\right]\right\} dV \\ &= ik\nabla' \times \int_{D^+} \mathbf{E}^r\left(\mathbf{r}\right)\delta\left(\mathbf{r},\mathbf{r}'\right)dV + ik[\nabla_i' \times \int_{D^+} \mathbf{E}^r\left(\mathbf{r}\right)\delta\left(\mathbf{r},\mathbf{r}_i'\right)dV] \cdot \mathbf{I}_i = k^2 Z \begin{cases} \mathbf{H}^r\left(\mathbf{r}'\right) & , \quad z'>0 \\ \mathbf{H}^r{}_i\left(\mathbf{r}_i'\right) & , \quad z'<0 \end{cases}. \end{aligned} \tag{2.10}$$

Substituting (2.7) and (2.9) in *I*, the right-hand side of (2.1), we find

$$I = \int_\sigma \left[\hat{n} \times \mathbf{E}^r(\mathbf{r})\right] \cdot \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')dS + ikZ\int_S \mathbf{H}^r(\mathbf{r}) \cdot \left[\hat{n} \times \boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')\right]dS\,. \tag{2.11}$$

Combining the last two results, we find that

$$\int_\sigma \left[\hat{z} \times \mathbf{E}^r(\mathbf{r})\right] \cdot \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')dS - ikZ\int_S \left[\hat{n} \times \mathbf{H}^r(\mathbf{r})\right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')dS = k^2 Z \begin{cases} \mathbf{H}^r\left(\mathbf{r}'\right) & , \quad z'>0 \\ \mathbf{H}^r{}_i\left(\mathbf{r}_i'\right) & , \quad z'<0 \end{cases}. \tag{2.12}$$

This is the integral representation of the magnetic field in the upper-half space. If we use the definition of the dyadic from (2.2), we can write it as

$$\begin{aligned} &2\int_\sigma \left[\hat{z} \times \mathbf{E}^r(\mathbf{r})\right] \cdot \left[\nabla\nabla g(\mathbf{r},\mathbf{r}') + k^2 g(\mathbf{r},\mathbf{r}')\mathrm{I}_t\right]dS \\ &-ikZ\int_S \left\{\left[\hat{n} \times \mathbf{H}^r(\mathbf{r})\right] \times \nabla g(\mathbf{r},\mathbf{r}') + \left(\left[\hat{n} \times \mathbf{H}^r(\mathbf{r})\right] \times \nabla g(\mathbf{r},\mathbf{r}_i')\right)_i\right\}dS = ikZ \begin{cases} \mathbf{H}^r\left(\mathbf{r}'\right) & , \quad z'>0 \\ \left[\mathbf{H}^r\left(\mathbf{r}_i'\right)\right]_i, & \quad z'<0 \end{cases}. \end{aligned} \tag{2.13}$$

3. INTEGRAL REPRESENTATIONS INSIDE THE COAXIAL LINE

We proceed to find a representation of the magnetic field inside the coaxial line. We apply Green's second identity to the region *D* of Figure 3.1. In the *z*-direction, this region extends between $z = -d$ and $z = 0$. In the lateral direction, it is bounded by the walls of the coaxial line. Thus, the four surfaces that bound *D* are τ, σ, S_a and S_b. The normal is directed into this region.

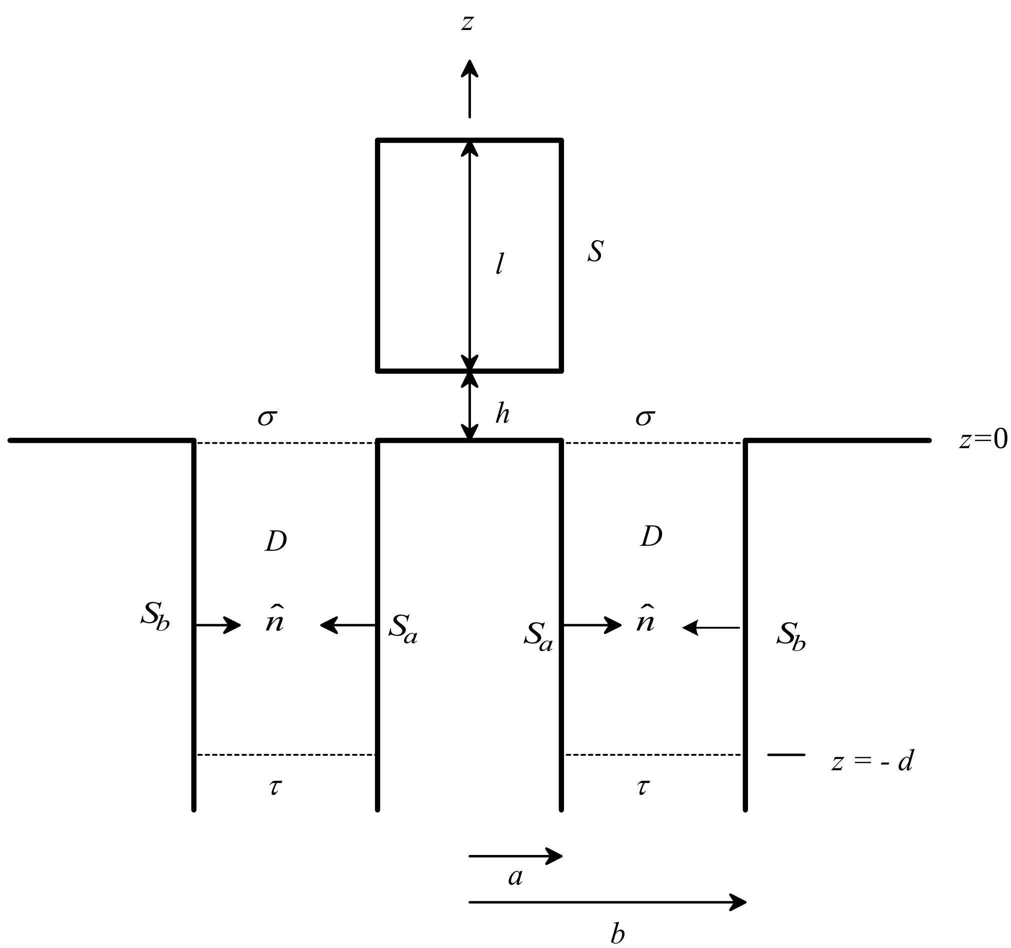

Figure 3.1. The region *D* of application of Green's second identity.

Let

$$\varpi = \sigma \cup \tau \cup S_a \cup S_b . \tag{3.1}$$

We can use the same identity as for (2.1) to obtain an integral representation for the total magnetic field in *D*; thus,

$$\int_D \left\{ \mathbf{E}^t(\mathbf{r}) \cdot \left[\nabla \times \nabla \times \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}') \right] - \left[\nabla \times \nabla \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}') \right\} dV$$

$$= \int_{\varpi} \left\{ \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \nabla \times \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') + \left[\hat{n} \times \nabla \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') \right\} dS \tag{3.2}$$

where $\mathbf{\Gamma}$ is the free-space dyadic Green's function, defined by

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}') = -ik\nabla \times \left[g(\mathbf{r},\mathbf{r}')\mathrm{I} \right] \tag{3.3}$$

and satisfying the differential equation

$$\nabla \times \nabla \times \mathbf{\Gamma}\left(\mathbf{r},\mathbf{r}'\right) - k^2 \Gamma\left(\mathbf{r},\mathbf{r}'\right) = ik\nabla \times \left[\delta(\mathbf{r},\mathbf{r}')\mathrm{I} \right]. \tag{3.4}$$

The total electric field in D satisfies the reduced wave equation (2.8) and the boundary condition

$$\hat{n} \times \mathbf{E}^t(\mathbf{r}) = \mathbf{0}\,, \quad \mathbf{r} \in S_a \cup S_b\,. \tag{3.5}$$

As with (2.1), the volume integral I_D in (3.2) yields

$$I_D = \begin{cases} k^2 Z\mathbf{H}^t(\mathbf{r}')\,, & \mathbf{r}' \in D \\ \mathbf{0}\,, & \mathbf{r}' \notin D \end{cases} \tag{3.6}$$

while the surface integral I_ϖ becomes

$$\begin{aligned} I_\varpi &= -ikZ \int_{S_a \cup S_b} \left\{ \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') \right\} dS \\ &\quad + \int_{\sigma \cup \tau} \left\{ \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \nabla \times \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') - ikZ\left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') \right\} dS\,. \end{aligned} \tag{3.7}$$

Combining the last two statements, we have that

$$\begin{aligned} &\int_{\sigma \cup \tau} \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \nabla \times \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') dS \\ &\qquad - ikZ\int_{\varpi} \left\{ \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \mathbf{\Gamma}(\mathbf{r},\mathbf{r}') \right\} dS = \begin{cases} k^2 Z\mathbf{H}^t(\mathbf{r}')\,, & \mathbf{r}' \in D \\ \mathbf{0}\,, & \mathbf{r}' \notin D \end{cases}. \end{aligned} \tag{3.8}$$

With the definition of the dyadic in (3.3), this becomes

$$\begin{aligned} &\int_{\sigma \cup \tau} \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \left[\nabla\nabla g(\mathbf{r},\mathbf{r}') + k^2 g(\mathbf{r},\mathbf{r}')\mathrm{I} \right] dS \\ &\qquad - ikZ\int_{\varpi} \left\{ \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \times \nabla g(\mathbf{r},\mathbf{r}') \right\} dS = \begin{cases} ikZ\mathbf{H}^t(\mathbf{r}')\,, & \mathbf{r}' \in D \\ \mathbf{0}\,, & \mathbf{r}' \notin D \end{cases}. \end{aligned} \tag{3.9}$$

This is the integral representation of the magnetic field in D.

4. ELIMINATION OF THE ELECTRIC FIELD

In these section, we eliminate one of the unknowns, the electric field, from the magnetic field integral representations.

When $z'>0$, we get from (3.9) that

$$\int_{\sigma}\left[\hat{z}\times\mathbf{E}^{t}(\mathbf{r})\right]\cdot\left[\nabla\nabla g(\mathbf{r},\mathbf{r}')+k^{2}g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS=\int_{\tau}\left[\hat{z}\times\mathbf{E}^{t}(\mathbf{r})\right]\cdot\left[\nabla\nabla g(\mathbf{r},\mathbf{r}')+k^{2}g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS$$
$$-ikZ\int_{\varpi}\left\{\left[\hat{n}\times\mathbf{H}^{t}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')\right\}dS\,,\quad z'>0\,. \tag{4.1}$$

Since the electric field is continuous in crossing σ, we can substitute this in the top line of (2.13) to get

$$2\int_{\tau}\left[\hat{z}\times\mathbf{E}^{t}(\mathbf{r})\right]\cdot\left[\nabla\nabla g(\mathbf{r},\mathbf{r}')+k^{2}g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS-i2kZ\int_{\varpi}\left\{\left[\hat{n}\times\mathbf{H}^{t}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')\right\}dS$$
$$-ikZ\int_{S}\left\{\left[\hat{n}\times\mathbf{H}^{r}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')+\left(\left[\hat{n}\times\mathbf{H}^{r}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}_{i}')\right)_{i}\right\}dS=ikZ\mathbf{H}^{r}\left(\mathbf{r}'\right),\quad \mathbf{r}'\in D^{+}\,. \tag{4.2}$$

In a similar way, we can substitute the bottom line of (2.13) in the top line of (3.9) to get

$$\int_{\tau}\left[\hat{z}\times\mathbf{E}^{t}(\mathbf{r})\right]\cdot\left[\nabla\nabla g(\mathbf{r},\mathbf{r}')+k^{2}g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS$$
$$-\frac{ikZ}{2}\int_{S}\left\{\left[\hat{n}\times\mathbf{H}^{r}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')+\left(\left[\hat{n}\times\mathbf{H}^{r}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}_{i}')\right)_{i}\right\}dS$$
$$-ikZ\int_{\varpi}\left\{\left[\hat{n}\times\mathbf{H}^{t}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')\right\}dS=ikZ\left\{\mathbf{H}^{t}(\mathbf{r}')+\frac{1}{2}\left[\mathbf{H}^{r}\left(\mathbf{r}_{i}'\right)\right]_{i}\right\},\quad \mathbf{r}'\in D\,. \tag{4.3}$$

These two expressions give us an integral representation of the magnetic field everywhere. The integrals over σ do not involve the electric field.

We can go a step further and eliminate the electric field altogether. On the annular disk τ, we have that $z=-d$. We can show that, as $d\to\infty$, the integral over τ tends to zero as d^{-1}. Taking this limit, we can write

$$-\int_{S}\left\{\left[\hat{n}\times\mathbf{H}^{r}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')+\left(\left[\hat{n}\times\mathbf{H}^{r}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}_{i}')\right)_{i}\right\}dS$$
$$-2\int_{\varpi}\left\{\left[\hat{n}\times\mathbf{H}^{t}(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')\right\}dS=\mathbf{H}^{r}\left(\mathbf{r}'\right)\,,\ \mathbf{r}'\in D^{+} \tag{4.4}$$

and

$$-\frac{1}{2}\int_S \left\{ \left[\hat{n}\times\mathbf{H}^r(\mathbf{r}) \right]\times\nabla g(\mathbf{r},\mathbf{r}') + \left(\left[\hat{n}\times\mathbf{H}^r(\mathbf{r}) \right]\times\nabla g(\mathbf{r},\mathbf{r}_i') \right)_i \right\} dS$$

$$-\int_{\varpi} \left\{ \left[\hat{n}\times\mathbf{H}^t(\mathbf{r}) \right]\times\nabla g(\mathbf{r},\mathbf{r}') \right\} dS = \mathbf{H}^t(\mathbf{r}') + \frac{1}{2}\left[\mathbf{H}^r\left(\mathbf{r}_i'\right) \right]_i , \quad \mathbf{r}'\in D . \qquad (4.5)$$

In the last two expressions, the region D extends over the entire space between the walls of the coaxial line.

For convenience, we define electric current densities on the various surfaces

$$\mathbf{J}_a(\mathbf{r}) = \hat{n}\times\mathbf{H}^t(\mathbf{r}) \text{ , } \mathbf{r}\in S_a \qquad (4.6)$$

$$\mathbf{J}_b(\mathbf{r}) = \hat{n}\times\mathbf{H}^t(\mathbf{r}) \text{ , } \mathbf{r}\in S_b \qquad (4.7)$$

$$\mathbf{J}_\sigma(\mathbf{r}) = -\hat{z}\times\mathbf{H}^t(\mathbf{r}) \text{ , } \mathbf{r}\in \sigma \qquad (4.8)$$

$$\mathbf{J}_S(\mathbf{r}) = \hat{n}\times\mathbf{H}^r(\mathbf{r}) \text{ , } \mathbf{r}\in S \qquad (4.9)$$

and substitute in (4.4) and (4.5)

$$-2\left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b} \mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS \right\}$$

$$-\int_S \left\{ \mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS = \mathbf{H}^r\left(\mathbf{r}'\right) \quad , \quad \mathbf{r}'\in D^+ \qquad (4.10)$$

$$-\left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b} \mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS \right\}$$

$$-\frac{1}{2}\int_S \left\{ \mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS = \mathbf{H}^t(\mathbf{r}') + \frac{1}{2}\mathbf{H}^r{}_i\left(\mathbf{r}_i'\right), \quad \mathbf{r}'\in D . \qquad (4.11)$$

These are the two integral representations for the magnetic field in terms of the unknown current densities. Equation (4.10) clearly displays the dependence of the radiated fields on the fields inside the coaxial line. Conversely, Equation (4.11) displays the dependence of the total fields inside the coaxial line on the geometry of the monopole and the resulting currents on it. Both expressions indicate the existence of a ground plane through the presence of the image of the gradient of the scalar Green’s function. We will use these integral representations next to obtain integral equations for the unknown current densities.

5. INTEGRAL EQUATIONS ON □ AND *S*

The derivation of integral equations for the four unknown current densities is based on a theorem of Müller (reference 4), Theorem 46, p. 205). If we have a region D bounded by a closed surface S, with the unit normal $\hat{n}$ pointing in the exterior (away from D), and if the surface current density $\mathbf{J}$ on S is continuous, then at any point $\mathbf{r}'$ of S (with normal $\hat{n}'$) we have that

$$\hat{n}' \times \int_{\substack{S_e \\ S_i}} \left[\mathbf{J}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')\right] dS = \mp \frac{1}{2}\mathbf{J}(\mathbf{r}') + \hat{n}' \times \int_S \left[\mathbf{J}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')\right] dS \tag{5.1}$$

where the expression on the left is to be interpreted in the sense of a limit, i.e., in the limit as we approach the surface from its exterior (S_e) or interior (S_i).

If we apply this theorem to (4.10), with the point $\mathbf{r}'$ on S and the approach from the exterior, we get

$$\begin{aligned} &-2\hat{n}' \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS + \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS \right\} \\ &-\hat{n}' \times \int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i')\right]_i \right\} dS = \frac{1}{2}\mathbf{J}_S\left(\mathbf{r}'\right), \quad \mathbf{r}' \in S . \end{aligned} \tag{5.2}$$

Again using (4.10), we approach a point on σ

$$\begin{aligned} &\hat{z}' \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS \right. \\ &\qquad \left. + \frac{1}{2}\int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i')\right]_i \right\} dS \right\} \\ &\qquad\quad + \hat{z}' \times \int_{\sigma} \left[\mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')\right] dS = \mathbf{J}_\sigma\left(\mathbf{r}'\right), \quad \mathbf{r}' \in \sigma . \end{aligned} \tag{5.3}$$

We note that we can also use (4.11) to obtain this last result. We also observe that

$$\nabla g(\mathbf{r},\mathbf{r}') = \nabla g(\mathbf{r},\mathbf{r}_i'), \quad z' = 0 \tag{5.4}$$

and that

$$\hat{z} \times \left[\mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')\right] = \mathbf{J}_\sigma(\mathbf{r}) \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} - \left[\hat{z} \cdot \mathbf{J}_\sigma(\mathbf{r})\right] \nabla g(\mathbf{r},\mathbf{r}') . \tag{5.5}$$

The last term here is zero because the current density does not have a z-component. Also, the derivative of the scalar Green's function with respect to z is equal to zero when $z = z' = 0$. Thus, in place of (5.3), we have

$$\hat{z} \times \left\{ \int_{S_a} \mathbf{J}_{S_a}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{S_b} \mathbf{J}_{S_b}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{S} \mathbf{J}_{S}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS \right\} = \mathbf{J}_{\sigma}(\mathbf{r}') \text{ , } \mathbf{r}' \in \sigma \tag{5.6}$$

We can also use (4.11) to obtain integral equations on the walls of the coaxial line. If the geometry is *exactly* as shown in Figure 2, then there is no problem. If, however, the geometry is more general, then we cannot use (4.11). As an example, consider the case where the radius of the monopole is greater than b. Then, for a point of S_a or S_b with z-coordinate between $-h$ and $-(h + l)$, the point $\mathbf{r}_i'$ is inside the monopole, where the radiated magnetic field is not defined. For this reason, we develop a new representation for the magnetic field inside the coaxial line.

6. A SECOND REPRESENTATION OF THE MAGNETIC FIELD INSIDE THE COAXIAL LINE

For reasons presented at the end of the last section, we proceed to obtain another representation of the magnetic field inside the coaxial line. In place of (3.2), we write

$$\int_D \left\{ \mathbf{E}^t(\mathbf{r}) \cdot \left[\nabla\times\nabla\times\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')\right] - \left[\nabla\times\nabla\times\mathbf{E}^t(\mathbf{r})\right]\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')\right\} dV$$
$$= \int_{\varpi} \left\{ -\mathbf{E}^t(\mathbf{r}) \cdot \left[\hat{n}\times\nabla\times\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')\right] + \left[\hat{n}\times\nabla\times\mathbf{E}^t(\mathbf{r})\right]\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')\right\} dS\,, \quad \mathbf{r}'\in D \tag{6.1}$$

where the Green's dyadic of the second kind is defined by

$$\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') = -ik\nabla\times\left[g(\mathbf{r},\mathbf{r}')\mathrm{I} - g(\mathbf{r},\mathbf{r}_i')\mathrm{I}_i\right] \tag{6.2}$$

and satisfies the differential equation

$$\nabla\times\nabla\times\boldsymbol{\Gamma}_2\left(\mathbf{r},\mathbf{r}'\right) - k^2\boldsymbol{\Gamma}_2\left(\mathbf{r},\mathbf{r}'\right) = ik\nabla\times\left[\delta(\mathbf{r},\mathbf{r}')\mathbf{I} - \delta(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i\right] \tag{6.3}$$

and the boundary condition

$$\hat{z}\times\nabla\times\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') = \mathbf{0} \;\text{ at } z=0\,. \tag{6.4}$$

In place of (3.6), we then have

$$I_D = k^2 Z\mathbf{H}^t(\mathbf{r}')\,, \quad \mathbf{r}'\in D \tag{6.5}$$

and, in place of (3.7),

$$I_{\varpi} = \int_{\varpi} \left\{ -\mathbf{E}^t(\mathbf{r}) \cdot \left[\hat{n}\times\nabla\times\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')\right] + \left[\hat{n}\times\nabla\times\mathbf{E}^t(\mathbf{r})\right]\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')\right\} dS\,. \tag{6.6}$$

As in Section 3, the surface integral over τ vanishes as $d\to\infty$. Because of (6.4), the first term vanishes over σ. It also vanishes over the walls of the coaxial line since the tangential component of the total electric field is zero there. We are thus left with

$$I_{\varpi} = -ikZ\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS + \int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS\right\}. \tag{6.7}$$

Combining (6.5) and (6.7), we find

$$\int_{S_a}\mathbf{J}_a(\mathbf{r})\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS + \int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\cdot\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS = ik\mathbf{H}^t(\mathbf{r}')\,, \quad \mathbf{r}'\in D \tag{6.8}$$

and, if we take definition (6.2) of the dyadic into consideration,

$$
\begin{aligned}
&-\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS-\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS \\
&\qquad -\int_{\sigma}\left\{\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS=\mathbf{H}^t(\mathbf{r}')\,,\quad \mathbf{r}'\in D\,.
\end{aligned}
\tag{6.9}
$$

But on σ

$$
\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i=2\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')\,,\quad z=0\,.
\tag{6.10}
$$

Thus,

$$
\begin{aligned}
&-\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS-\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS \\
&\qquad -2\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS=\mathbf{H}^t(\mathbf{r}')\,,\quad \mathbf{r}'\in D\,.
\end{aligned}
\tag{6.11}
$$

This is the second representation of the magnetic field inside the coaxial line.

7. INTEGRAL EQUATIONS ON THE WALLS OF THE COAXIAL LINE

In this section, we use the integral representation of the magnetic field that we obtained in the last section to get integral equations on the semi-infinite walls of the coaxial line. To this end, we employ (5.1) and (6.11). In the approach to the inner wall we get

$$\begin{aligned}
&-\hat{n}'\times\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\qquad-\hat{n}'\times\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\qquad-2\hat{n}'\times\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS=\frac{1}{2}\mathbf{J}_a(\mathbf{r}')\,,\quad \mathbf{r}'\in S_a
\end{aligned}\tag{7.1}$$

while in the approach to the outer

$$\begin{aligned}
&-\hat{n}'\times\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\qquad-\hat{n}'\times\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\qquad-2\hat{n}'\times\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS=\frac{1}{2}\mathbf{J}_b(\mathbf{r}')\,,\quad \mathbf{r}'\in S_b\,.
\end{aligned}\tag{7.2}$$

Following this approach, we can ask what happens to (6.11) as the observation point $\mathbf{r}'$ approaches σ. If we can get an integral equation on σ, then, together with (7.1) and (7.2), we will have three equations in three unknowns and, thus, we will be able to solve for the three current densities. We note, however, that none of these three equations contains information about the upper-half space. This says that the geometry of the upper-half space does not influence the behavior of these currents, which does not seem to be correct. We proceed to perform this calculation. First, we cross (6.11) with the unit normal on σ

$$\begin{aligned}
&\int_{S_a}\hat{z}\times\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right\}dS\\
&\qquad+\int_{S_b}\hat{z}\times\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right\}dS\\
&\qquad+2\int_{\sigma}\hat{z}\times\left[\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')\right]dS=\mathbf{J}_\sigma(\mathbf{r}')\,,\quad \mathbf{r}'\in D\,.
\end{aligned}\tag{7.3}$$

In the first integral above, only the components transverse to the z-axis are involved; hence, we can write instead

$$\int_{S_a}\hat{z}\times\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right\}dS$$

$$= \int_{S_a} \left\{ \mathbf{J}_a(\mathbf{r}) \hat{z} \cdot \nabla \left[g(\mathbf{r},\mathbf{r}') - g(\mathbf{r},\mathbf{r}_i') \right] - \left[\hat{z} \cdot \mathbf{J}_a(\mathbf{r}) \right] \nabla \left[g(\mathbf{r},\mathbf{r}') - g(\mathbf{r},\mathbf{r}_i') \right] \right\} dS \tag{7.4}$$

which is equal to zero because of (5.4). The same is true about the second integral in (7.3). As the observation point approaches the surface, we apply (5.1) to the remaining integral to get that

$$\int_{\sigma} \hat{z} \times \left[\mathbf{J}_{\sigma}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') \right] dS = \mathbf{0}\,, \quad \mathbf{r}' \in \sigma \tag{7.5}$$

with the last statement also following from (5.5). Thus, we have the identity $0 = 0$ or that we cannot obtain an integral equation from (6.11) when the observation point is on the surface σ.

8. THE SYSTEM OF INTEGRAL EQUATIONS

We have four unknowns, namely, the electric current densities defined in (4.6) through (4.9). With these, we associate the integral equations (5.2), (5.6), (7.1), and (7.2). A drawback of this system is that the surfaces S_a and S_b are semi-infinite in the z-direction and, hence, they will have to be terminated in any typical boundary-element scheme. For this reason, in Part 1I of this study, we return to the scheme where the parameter d is finite and see whether we can expand the system of integral equations to accommodate a magnetic current density.

One remark we wish to make is that (5.6) can also be obtained using the approach suggested by Hansen and Yaghjian (reference 5). Here, the region of application of Green's second identity is the interior of the coaxial line and the entire upper-half space except for the region occupied by the monopole (Figure 3.1). If, with it, we use the free-space Green's function in (3.3) we get

$$-\int_{\varpi} \mathbf{J}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS = \mathbf{H}(\mathbf{r}') , \quad \mathbf{r}' \in D \cup D^+ , \quad \varpi = S_a \cup S_b \cup \sigma^c \cup S \tag{8.1}$$

where σ^c stands for the metallic part of the xy-plane. Writing this compact statement out, we have

$$-\int_{\sigma^c} \mathbf{J}_{\sigma^c}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS - \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS - \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS$$
$$-\int_{S} \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS = \begin{cases} \mathbf{H}^r(\mathbf{r}') , & \mathbf{r}' \in D^+ \\ \mathbf{H}^t(\mathbf{r}') , & \mathbf{r}' \in D \end{cases} \tag{8.2}$$

where

$$\mathbf{J}_{\sigma^c}(\mathbf{r}) = \hat{z} \times \mathbf{H}^r(\mathbf{r}) , \quad \mathbf{r} \in \sigma^c . \tag{8.3}$$

In passing, we mention that, in (8.1), there should also be present an integral over a hemisphere with center the origin and a radius that tends to infinity. In the limit, however, this integral vanishes because the radiated fields obey the Silver-Müller radiation conditions. If this were a problem in scattering by a plane wave, as in reference 3, then we would have to use a stationery phase method to show that the integral vanishes. This is because a plane wave does not satisfy the Silver-Müller radiation conditions. Besides the scattered fields, the sum of the fields of the incident and reflected plane waves would appear in this integral.

If $\mathbf{r} \in \sigma$ and we pre-cross (8.2) with the unit normal along the z-axis, then the surface integral over σ^c becomes zero for the same reason as (5.5), and we get

$$\hat{z} \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S} \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}')dS \right\} = \mathbf{J}_\sigma(\mathbf{r}') , \mathbf{r}' \in \sigma \tag{8.4}$$

which is identical with (5.6). We note that this is the only occasion in which the surface integral over the metallic part of the *xy*-plane is zero; otherwise, (8.2) could be used to obtain integral equations over the whole structure.

Another remark is that, in the formulation resulting in (8.1), instead of the free-space Green's dyadic we can use the one in (2.2). Because of the property (2.7), the integral representation will not involve a surface integral over σ^c. Proceeding as in Section 2, we get in place of (2.13)

$$\begin{aligned}
&-\int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS \\
&\quad - \int_{S_a} \left\{ \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS \\
&\quad - \int_{S_b} \left\{ \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS = \mathbf{H}\left(\mathbf{r}'\right) + \mathbf{H}_i\left(\mathbf{r}_i'\right), \quad \mathbf{r}' \in D^+ \cup D
\end{aligned} \tag{8.5}$$

where **H** stands for the total magnetic field in either of the two regions. This statement is not entirely correct. For example, if $\mathbf{r}' = (\rho' > b, \varphi', z' > 0)$, then $\mathbf{r}_i'$ does not belong to the region of integration *D* and the last term in (8.5) should be equal to zero. We can correct for this by introducing characteristic functions and writing a separate statement for each of the two regions. For example, in the upper region

$$\begin{aligned}
&-\int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS \\
&\quad - \int_{S_a} \left\{ \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS \\
&\quad - \int_{S_b} \left\{ \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS = \mathbf{H}^r\left(\mathbf{r}'\right) + \chi_D\left(\mathbf{r}_i'\right)\left[\mathbf{H}^t\left(\mathbf{r}_i'\right) \right]_i, \quad \mathbf{r}' \in D^+
\end{aligned} \tag{8.6}$$

where χ_D, the characteristic function of the region *D*, is defined by

$$\chi_D\left(\mathbf{r}\right) = \begin{cases} 1, & \mathbf{r} \in D \\ 0, & \mathbf{r} \notin D \end{cases}. \tag{8.7}$$

Still, we have to worry about the image point being on the boundary of *D* (the walls of the coaxial line). We can write a statement similar to (8.6) when the observation point is in *D*, with the same concerns.

The problem with an expression like (8.6) becomes evident when we try to obtain integral equations. The image of a point of *S* may correspond to a point somewhere in *D*. Thus, we may generate unknowns not only on the metallic boundaries but, also, in the regions *D* and D^+. This is quite unacceptable and we abandon this approach.

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CHAPTER 2
EQUATIONS EXTENDING OVER PART OF THE WALLS OF THE LINE

1. INTRODUCTION

In Chapter 1, we derived BIEs for the geometry of Figure 1.1. All surfaces are perfectly conducting. The ground plane is infinite and the coaxial line extends to infinity in the lower-half space. The inner and outer radii of the transmission line are a and b, respectively. The line supports only a TEM wave which means that (references 1 and 2

$$k\frac{a+b}{2}<1 \tag{1.1}$$

where k is the wavenumber of the time-harmonic ($\mathrm{e}^{+i\omega t}$) electromagnetic wave in the line.

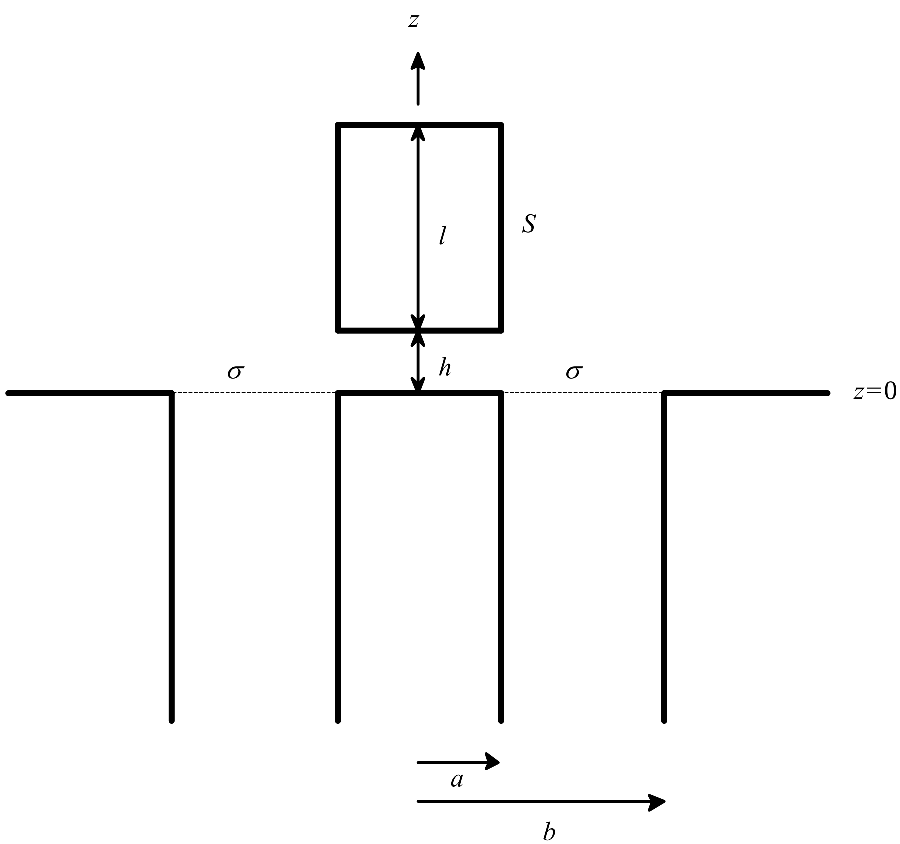

Figure 1.1. A cylindrical monopole over a ground plane, fed by a coaxial line.

The excitation of the line occurs at $z=-\infty$ and results in a TEM wave with fields

$$\mathbf{E}^g(\mathbf{r})=\frac{V\,\mathrm{e}^{-ikz}}{\ln\left(\frac{b}{a}\right)\rho}\hat{\rho}\,,\quad \mathbf{H}^g(\mathbf{r})=\frac{YV\,\mathrm{e}^{-ikz}}{\ln\left(\frac{b}{a}\right)\rho}\hat{\varphi}\,. \tag{1.2}$$

where V is the voltage of the inner conductor with respect to the outer, and Y is the free-space admittance. We have also employed cylindrical coordinates (ρ,φ,z). Since the line is not infinite, we also have an induced wave in the line with fields $\{\mathbf{E}^i, \mathbf{H}^i\}$. The total fields $\{\mathbf{E}^t, \mathbf{H}^t\}$ in the coaxial line are the sum of the generator and induced fields

$$\mathbf{E}^t = \mathbf{E}^g + \mathbf{E}^i \,, \quad \mathbf{H}^t = \mathbf{H}^g + \mathbf{H}^i \,. \tag{1.3}$$

In the upper-half space we have radiated fields $\{\mathbf{E}^r, \mathbf{H}^r\}$.

The integral equations we derived in Chapter 1 on the walls of the coaxial line, extend over the entire (infinite) length of the line. Here we will develop a system of integral equations that will extend over a portion of the walls of the line. The region under consideration is shown in Figure 1.2. Inside the line, we will develop integral equations extending no deeper than $z = -d$. In the process, we will use a number of results of Chapter 1. Equation numbers from there will be preceded by the Roman numeral I.

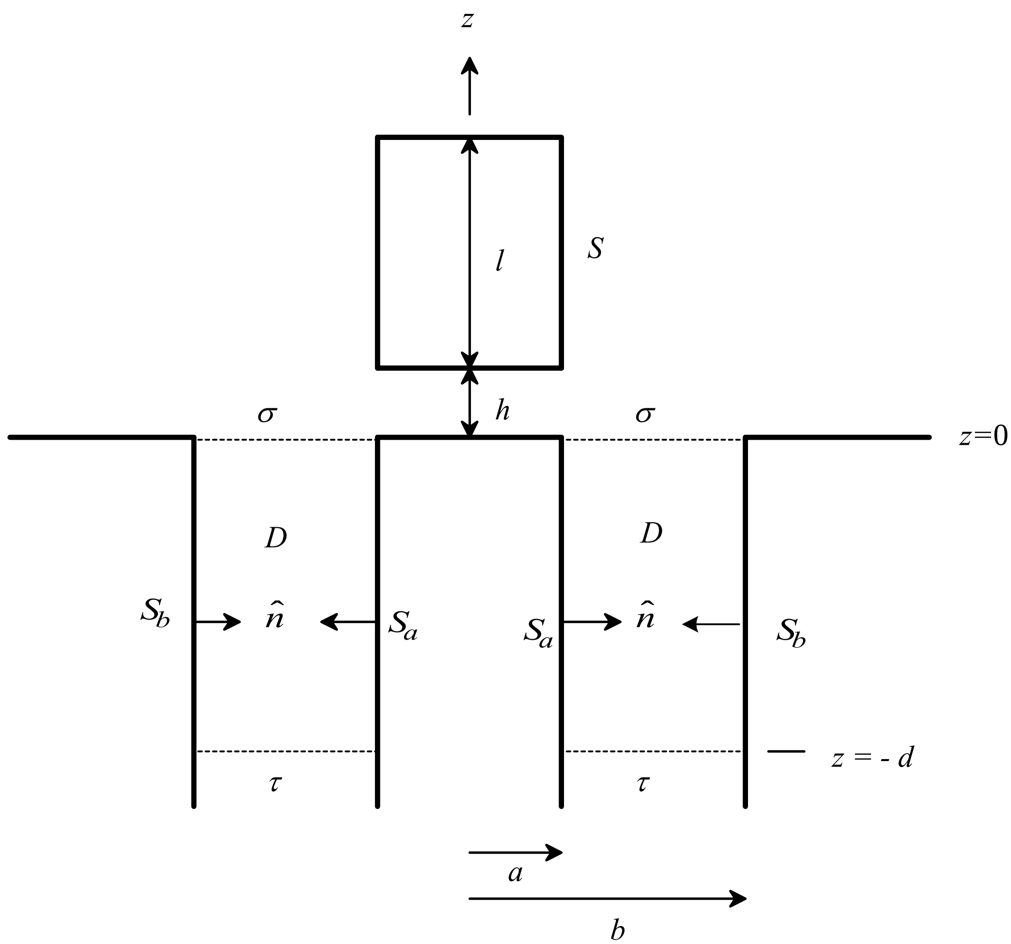

Figure 1.2. The region D of application of Green's second identity.

2. INTEGRAL REPRESENTATIONS IN THE EXTERIOR AND INTERIOR

For convenience, we define electric and magnetic current densities on τ by

$$\mathbf{J}_\tau(\mathbf{r}) = \hat{z}\times\mathbf{H}^t(\mathbf{r}) \quad , \quad \mathbf{M}_\tau(\mathbf{r}) = -\hat{z}\times\mathbf{E}^t(\mathbf{r}) \quad , \quad \mathbf{r}\in\tau \tag{2.1}$$

From (I.4.2)

$$2\int_\tau \left[\hat{z}\times\mathbf{E}^t(\mathbf{r})\right]\cdot\left[\nabla\nabla g(\mathbf{r},\mathbf{r}') + k^2 g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS - i2kZ\int_\varpi \left\{\left[\hat{n}\times\mathbf{H}^t(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}')\right\}dS$$
$$-ikZ\int_S \left\{\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS = ikZ\mathbf{H}^r\left(\mathbf{r}'\right), \quad \mathbf{r}'\in D^+ . \tag{2.2}$$

The surface ϖ is defined in (I.3.1). Taking into consideration the current density definitions (I.4.6)-(I.4.9) and the ones above, we write

$$-2\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\left[\nabla\nabla g(\mathbf{r},\mathbf{r}') + k^2 g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS$$
$$-i2kZ\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_\sigma \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_\tau \mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}$$
$$-ikZ\int_S \left\{\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS = ikZ\mathbf{H}^r\left(\mathbf{r}'\right), \quad \mathbf{r}'\in D^+ . \tag{2.3}$$

We next complete (I.6.7) by supplying the integral over τ from (I.6.6)

$$I_\varpi = -\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\nabla\times\mathbf{\Gamma}_2(\mathbf{r},\mathbf{r}')dS$$
$$-ikZ\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS\right.$$
$$\left. + \int_\sigma \mathbf{J}_\sigma(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS + \int_\tau \mathbf{J}_\tau(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS\right\} \tag{2.4}$$

Using (I.6.5) together with this, we get

$$-ikZ\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS + \int_\sigma \mathbf{J}_\sigma(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS + \int_\tau \mathbf{J}_\tau(\mathbf{r})\cdot\Gamma_2(\mathbf{r},\mathbf{r}')dS\right\}$$
$$-\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\nabla\times\mathbf{\Gamma}_2(\mathbf{r},\mathbf{r}')dS = k^2 Z\mathbf{H}^t(\mathbf{r}'), \quad \mathbf{r}'\in D . \tag{2.5}$$

By the definition (I.6.2) of the dyadic, the last integral in this becomes

$$\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\nabla\times\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS = -ik\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\left\{\nabla\nabla g(\mathbf{r},\mathbf{r}') + k^2 g(\mathbf{r},\mathbf{r}')\mathbf{I} - \nabla\left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i - k^2 g(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i\right\}dS$$

$$= -ik\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\nabla\left\{\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS - ik^3\int_\tau\left\{g(\mathbf{r},\mathbf{r}')\mathbf{M}_\tau(\mathbf{r}) - g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{M}_\tau(\mathbf{r})\right]_i\right\}dS$$

(2.6)

For the first integral on the right, we use the divergence theorem to write

$$\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\nabla\left\{\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS$$

$$= \int_\tau\left\{\nabla\cdot\left[\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)\right] - \nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)\right\}dS$$

$$= \int_\gamma \hat{\nu}\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)ds - \int_\tau \nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)dS\,. \qquad (2.7)$$

The contour γ is composed of the two contours that constitute the boundary of τ. The unit normal $\hat{\nu}$ on γ is equal to the negative of the normal on each of the two walls of the coaxial line. The component of the magnetic current density along it is, by (2.1), the tangential component of the total electric field on each wall. Its value there is zero. Thus, the contour integral is equal to zero, and we have that

$$\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\nabla\times\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}')dS$$

$$= ik\int_\tau \nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)dS - ik^3\int_\tau\left\{g(\mathbf{r},\mathbf{r}')\mathbf{M}_\tau(\mathbf{r}) - g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{M}_\tau(\mathbf{r})\right]_i\right\}dS\,. \qquad (2.8)$$

Substitution of this and (I.6.11) in (2.5) gives

$$-\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS - \int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS$$
$$-2\int_\sigma \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS - \int_\tau\left\{\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS$$
$$+\frac{iY}{k}\int_\tau \nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}') - \left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)dS - ikY\int_\tau\left\{g(\mathbf{r},\mathbf{r}')\mathbf{M}_\tau(\mathbf{r}) - g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{M}_\tau(\mathbf{r})\right]_i\right\}dS$$
$$= \mathbf{H}^t(\mathbf{r}')\,, \quad \mathbf{r}'\in D\,. \qquad (2.9)$$

This is the total magnetic field representation in D. We proceed to get one for the total electric field. We begin with Green's second identity

$$\int_D\left\{\mathbf{H}^t(\mathbf{r})\cdot\left[\nabla\times\nabla\times\boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')\right] - \left[\nabla\times\nabla\times\mathbf{H}^t(\mathbf{r})\right]\cdot\boldsymbol{\Gamma}_1(\mathbf{r},\mathbf{r}')\right\}dV$$

$$= \int_{\varpi} \left\{ \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \nabla \times \mathbf{\Gamma}_1(\mathbf{r},\mathbf{r}') + ikY \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \mathbf{\Gamma}_1(\mathbf{r},\mathbf{r}') \right\} dS \,, \quad \mathbf{r}' \in D \,. \tag{2.10}$$

The Green's dyadic of the first kind is defined in (I.2.2). For the volume integral, we get

$$I_D = -k^2 Y \mathbf{E}^t(\mathbf{r}') \,, \quad \mathbf{r}' \in D \,. \tag{2.11}$$

From (I.2.2)

$$\begin{aligned}
&\int_{\varpi} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \nabla \times \Gamma_1(\mathbf{r},\mathbf{r}') dS \\
&= -ik \int_{\varpi} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \nabla \left[\nabla g(\mathbf{r},\mathbf{r}') + \left(\nabla g(\mathbf{r},\mathbf{r}_i') \right)_i \right] dS \\
&\qquad\qquad -ik^3 \int_{\varpi} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \left[g(\mathbf{r},\mathbf{r}')\mathrm{I} + g(\mathbf{r},\mathbf{r}_i')\mathrm{I}_i \right] dS \\
&= ik \int_{\varpi} \nabla \cdot \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \left[\nabla g(\mathbf{r},\mathbf{r}') + \left(\nabla g(\mathbf{r},\mathbf{r}_i') \right)_i \right] dS \,. \\
&\qquad\qquad -ik^3 \int_{\varpi} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \left[g(\mathbf{r},\mathbf{r}')\mathrm{I} + g(\mathbf{r},\mathbf{r}_i')\mathrm{I}_i \right] dS \,.
\end{aligned} \tag{2.12}$$

The surface divergence theorem has been applied above to the closed surface ϖ. The result is, of course, zero. For the remaining term in (2.10)

$$\begin{aligned}
\int_{\varpi} \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \Gamma_1(\mathbf{r},\mathbf{r}') dS &= -\int_{\tau} \mathbf{M}_\tau(\mathbf{r}) \cdot \Gamma_1(\mathbf{r},\mathbf{r}') dS = ik \int_{\tau} \mathbf{M}_\tau(\mathbf{r}) \cdot \nabla \times \left[g(\mathbf{r},\mathbf{r}')\mathrm{I} + g(\mathbf{r},\mathbf{r}_i')\mathrm{I}_i \right] dS \\
&= ik \int_{\tau} \left\{ \mathbf{M}_\tau(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{M}_\tau(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS \,.
\end{aligned} \tag{2.13}$$

In arriving at this result, we have taken into consideration the fact that the total electric field is zero on the walls of the coaxial line and that the tangential component of the dyadic is equal to zero on the $z = 0$ plane. Substituting the above results in (2.10), we get

$$\begin{aligned}
&-\frac{iZ}{k} \int_{\varpi} \nabla \cdot \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \left[\nabla g(\mathbf{r},\mathbf{r}') + \left(\nabla g(\mathbf{r},\mathbf{r}_i') \right)_i \right] dS + ikZ \int_{\varpi} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \left[g(\mathbf{r},\mathbf{r}')\mathrm{I} + g(\mathbf{r},\mathbf{r}_i')\mathrm{I}_i \right] dS \\
&+ \int_{\tau} \left\{ \mathbf{M}_\tau(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{M}_\tau(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS = \mathbf{E}^t(\mathbf{r}') \,, \quad \mathbf{r}' \in D \,.
\end{aligned} \tag{2.14}$$

Moreover, breaking the first and second integrals into their constituent parts

$$-\frac{iZ}{k} \left\{ \int_{S_a} \nabla \cdot \left[\mathbf{J}_a(\mathbf{r}) \right] \left[\nabla g(\mathbf{r},\mathbf{r}') + \left(\nabla g(\mathbf{r},\mathbf{r}_i') \right)_i \right] dS + \int_{S_b} \nabla \cdot \left[\mathbf{J}_b(\mathbf{r}) \right] \left[\nabla g(\mathbf{r},\mathbf{r}') + \left(\nabla g(\mathbf{r},\mathbf{r}_i') \right)_i \right] dS \right.$$

$$
\begin{aligned}
&+\int_{\tau} \nabla \cdot \left[\mathbf{J}_{\tau}(\mathbf{r})\right]\left[\nabla g(\mathbf{r},\mathbf{r}') + \left(\nabla g(\mathbf{r},\mathbf{r}_i')\right)_i\right] dS + 2\int_{\sigma} \nabla \cdot \left[\mathbf{J}_{\sigma}(\mathbf{r})\right]\nabla g(\mathbf{r},\mathbf{r}') dS \Bigg\} \\
&+ikZ\left\{\int_{S_a}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{J}_a(\mathbf{r}) + g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{J}_a(\mathbf{r})\right]_i\right\} dS + \int_{S_b}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{J}_b(\mathbf{r}) + g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{J}_b(\mathbf{r})\right]_i\right\} dS\right. \\
&+\int_{\tau}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{J}_{\tau}(\mathbf{r}) + g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{J}_{\tau}(\mathbf{r})\right]_i\right\} dS + 2\int_{\sigma} g(\mathbf{r},\mathbf{r}')\mathbf{J}_{\sigma}(\mathbf{r}) dS \Bigg\} \\
&+\int_{\tau}\left\{\mathbf{M}_{\tau}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{M}_{\tau}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\} dS = \mathbf{E}^t(\mathbf{r}')\,, \quad \mathbf{r}' \in D\,. \qquad (2.15)
\end{aligned}
$$

This is the integral representation for the electric field.

3. INTEGRAL EQUATIONS ON □

From (I.5.1) and (2.10), we get that

$$
\begin{aligned}
&-\hat{z}\times\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&-\hat{z}\times\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS-2\hat{z}\times\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\\
&-\hat{z}\times\int_{\tau}\left\{\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&+\frac{iY}{k}\hat{z}\times\int_{\tau}\nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}')-\left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)dS\\
&-ikY\hat{z}\times\int_{\tau}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{M}_\tau(\mathbf{r})-g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{M}_\tau(\mathbf{r})\right]_i\right\}dS=\frac{1}{2}\mathbf{J}_\tau(\mathbf{r}')\,,\ \mathbf{r}'\in\tau\,.
\end{aligned}
\tag{3.1}
$$

We can simplify this expression considerably. For the third term, we write

$$
-2\hat{z}\times\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS=-2\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}dS
\tag{3.2}
$$

while for the fourth

$$
\begin{aligned}
&-\hat{z}\times\left\{\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}=-\hat{z}\times\left[\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')\right]\\
&+\hat{z}\times\left\{\left[\hat{z}\times\mathbf{H}^t(\mathbf{r})\right]\times\nabla g(\mathbf{r},\mathbf{r}_i')\right\}_i=-\mathbf{J}_\tau(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}+\hat{z}\times\left\{\mathbf{H}^t(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial z}\right\}_i\\
&=-\mathbf{J}_\tau(\mathbf{r})\left[\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}-\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial z}\right]=\mathbf{J}_\tau(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial z}\,.
\end{aligned}
\tag{3.3}
$$

Thus, in place of (3.1), we can write

$$
\begin{aligned}
&-\hat{z}\times\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&-\hat{z}\times\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS-2\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}dS\\
&+\int_{\tau}\mathbf{J}_\tau(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial z}dS+\frac{iY}{k}\int_{\tau}\nabla\cdot\mathbf{M}_\tau(\mathbf{r})\hat{z}\times\left(\nabla g(\mathbf{r},\mathbf{r}')-\nabla g(\mathbf{r},\mathbf{r}_i')\right)dS\\
&-ikY\hat{z}\times\int_{\tau}\mathbf{M}_\tau(\mathbf{r})\left[g(\mathbf{r},\mathbf{r}')-g(\mathbf{r},\mathbf{r}_i')\right]dS=\frac{1}{2}\mathbf{J}_\tau(\mathbf{r}')\,,\ \mathbf{r}'\in\tau\,.
\end{aligned}
\tag{3.4}
$$

This is the integral equation for the electric current density on τ. For the magnetic current density, we have from (2.15)

$$-\frac{iZ}{k}\hat{z}\times\left\{\int_{S_a}\nabla\cdot\left[\mathbf{J}_a(\mathbf{r})\right]\left[\nabla g(\mathbf{r},\mathbf{r}')+\left(\nabla g(\mathbf{r},\mathbf{r}_i'))\right)_i\right]dS+\int_{S_b}\nabla\cdot\left[\mathbf{J}_b(\mathbf{r})\right]\left[\nabla g(\mathbf{r},\mathbf{r}')+\left(\nabla g(\mathbf{r},\mathbf{r}_i')\right)_i\right]dS\right.$$
$$\left.+\int_\tau\nabla\cdot\left[\mathbf{J}_\tau(\mathbf{r})\right]\left[\nabla g(\mathbf{r},\mathbf{r}')+\left(\nabla g(\mathbf{r},\mathbf{r}_i')\right)_i\right]dS+2\int_\sigma\nabla\cdot\left[\mathbf{J}_\sigma(\mathbf{r})\right]\nabla g(\mathbf{r},\mathbf{r}')dS\right\}$$
$$+ikZ\hat{z}\times\left\{\int_{S_a}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{J}_a(\mathbf{r})+g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{J}_a(\mathbf{r})\right]_i\right\}dS+\int_{S_b}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{J}_b(\mathbf{r})+g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{J}_b(\mathbf{r})\right]_i\right\}dS\right.$$
$$\left.+\int_\tau\left\{g(\mathbf{r},\mathbf{r}')\mathbf{J}_\tau(\mathbf{r})+g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{J}_\tau(\mathbf{r})\right]_i\right\}dS+2\int_\sigma g(\mathbf{r},\mathbf{r}')\mathbf{J}_\sigma(\mathbf{r})dS\right\}$$
$$+\hat{z}\times\int_\tau\left\{\mathbf{M}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')+\left[\mathbf{M}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS=-\frac{1}{2}\mathbf{M}_\tau(\mathbf{r}'),\quad \mathbf{r}'\in\tau. \tag{3.5}$$

We can re-write this as

$$-\frac{iZ}{k}\left\{\int_{S_a}\nabla\cdot\left[\mathbf{J}_a(\mathbf{r})\right]\hat{z}\times\left[\nabla g(\mathbf{r},\mathbf{r}')+\nabla g(\mathbf{r},\mathbf{r}_i')\right]dS+\int_{S_b}\nabla\cdot\left[\mathbf{J}_b(\mathbf{r})\right]\hat{z}\times\left[\nabla g(\mathbf{r},\mathbf{r}')+\nabla g(\mathbf{r},\mathbf{r}_i')\right]dS\right.$$
$$\left.+\int_\tau\nabla\cdot\left[\mathbf{J}_\tau(\mathbf{r})\right]\hat{z}\times\left[\nabla g(\mathbf{r},\mathbf{r}')+\nabla g(\mathbf{r},\mathbf{r}_i')\right]dS+2\int_\sigma\nabla\cdot\left[\mathbf{J}_\sigma(\mathbf{r})\right]\hat{z}\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}$$
$$+ikZ\left\{\int_{S_a}\hat{z}\times\mathbf{J}_a(\mathbf{r})\left[g(\mathbf{r},\mathbf{r}')+g(\mathbf{r},\mathbf{r}_i')\right]dS+\int_{S_b}\hat{z}\times\mathbf{J}_b(\mathbf{r})\left[g(\mathbf{r},\mathbf{r}')+g(\mathbf{r},\mathbf{r}_i')\right]dS\right.$$
$$\left.+\int_\tau\hat{z}\times\mathbf{J}_\tau(\mathbf{r})\left[g(\mathbf{r},\mathbf{r}')+g(\mathbf{r},\mathbf{r}_i')\right]dS+2\int_\sigma g(\mathbf{r},\mathbf{r}')\hat{z}\times\mathbf{J}_\sigma(\mathbf{r})dS\right\}$$
$$+\int_\tau\mathbf{M}_\tau(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial z}dS=-\frac{1}{2}\mathbf{M}_\tau(\mathbf{r}'),\quad \mathbf{r}'\in\tau. \tag{3.6}$$

4. SYSTEM OF INTEGRAL EQUATIONS

Besides the integral equations (3.4) and (3.6), the system of equations will include equations from the rest of the structures that make up the problem. Using the procedures of Chapter 1, we get from (2.9) that

$$
\begin{aligned}
&-\hat{n}'\times\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\quad-\hat{n}'\times\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS-2\hat{n}'\times\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\\
&\quad-\hat{n}'\times\int_{\tau}\left\{\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\quad+\frac{iY}{k}\hat{n}'\times\int_{\tau}\nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}')-\left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)dS\\
&\quad-ikY\hat{n}'\times\int_{\tau}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{M}_\tau(\mathbf{r})-g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{M}_\tau(\mathbf{r})\right]_i\right\}dS=\frac{1}{2}\mathbf{J}_a(\mathbf{r}')\,,\quad \mathbf{r}'\in S_a
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&-\hat{n}'\times\int_{S_a}\left\{\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\quad-\hat{n}'\times\int_{S_b}\left\{\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS-2\hat{n}'\times\int_{\sigma}\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\\
&\quad-\hat{n}'\times\int_{\tau}\left\{\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')-\left[\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS\\
&\quad+\frac{iY}{k}\hat{n}'\times\int_{\tau}\nabla\cdot\mathbf{M}_\tau(\mathbf{r})\left(\nabla g(\mathbf{r},\mathbf{r}')-\left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right)dS\\
&\quad-ikY\hat{n}'\times\int_{\tau}\left\{g(\mathbf{r},\mathbf{r}')\mathbf{M}_\tau(\mathbf{r})-g(\mathbf{r},\mathbf{r}_i')\left[\mathbf{M}_\tau(\mathbf{r})\right]_i\right\}dS=\frac{1}{2}\mathbf{J}_b(\mathbf{r}')\,,\quad \mathbf{r}'\in S_b\,.
\end{aligned}
\tag{4.2}
$$

From (I.4.2)

$$
\begin{aligned}
&ikY\hat{z}\times\int_{\tau}\mathbf{M}_\tau(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS+\frac{iY}{k}\hat{z}\times\int_{\tau}\mathbf{M}_\tau(\mathbf{r})\cdot\nabla\nabla g(\mathbf{r},\mathbf{r}')dS\\
&\quad-\hat{z}\times\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS+\int_{S_b}\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS+\int_{\tau}\mathbf{J}_\tau(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}\\
&\quad-\hat{z}\times\int_{S}\mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS=-\mathbf{J}_\sigma\left(\mathbf{r}'\right)\quad,\quad \mathbf{r}'\in\sigma
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
&2ikY\hat{n}' \times \int_{\tau} \mathbf{M}_{\tau}(\mathbf{r}) g(\mathbf{r},\mathbf{r}') dS + 2\frac{iY}{k}\hat{n}' \times \int_{\tau} \mathbf{M}_{\tau}(\mathbf{r}) \cdot \nabla\nabla g(\mathbf{r},\mathbf{r}') dS \\
&\quad - 2\hat{n}' \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS \right. \\
&\qquad\qquad \left. + \int_{\sigma} \mathbf{J}_{\sigma}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS + \int_{\tau} \mathbf{J}_{\tau}(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') dS \right\} \\
&\quad - \hat{n}' \times \int_{S} \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r},\mathbf{r}_i') \right]_i \right\} dS = \frac{1}{2}\mathbf{J}_S(\mathbf{r}'), \quad \mathbf{r}' \in S\,. \qquad (4.4)
\end{aligned}
$$

5. REMARKS

We have some concern here as to whether the equations we just derived are the right ones for determining the unknown currents. We will not know the answer to this until we compare numerical results between this system and the one of Chapter 1. If this system provides the correct answers, then it is the preferred one because it also provides substantial computational savings if we keep d small. We note that, in this formulation, current densities are assumed piecewise differentiable in the interior of the line. In Chapter 1, it is sufficient that these currents are continuous.

Equations (2.5) and (2.14) give the fields in the interior of the coaxial line. They are the result of applying Green's second identity in the region bounded by the closed surface ϖ, defined in (I.3.1)

$$\varpi = \sigma \cup \tau \cup S_a \cup S_b . \tag{5.1}$$

We make two remarks. The first is that the integral over the surface τ ($z = -d$) can be converted to an integral over the two cylindrical surfaces that join this surface to the identical disk at $z = -\infty$. Indeed, the walls and the two disks make up a closed surface. Application of the appropriate Green's second identity (in each of the two cases), with the observation point outside this surface ($-d < z' < 0$), leads to the result that the integral over τ is equal to the integral over the other three surfaces. As remarked in I, the integral over the disk at infinity is equal to zero; hence, the conclusion that the integral over τ is equal to the same integral over the cylindrical walls extending from – d to $-\infty$. In the process, the terms involving the circumferential components of the electric field drop out, because the walls are perfectly conducting, and we end-up with the same representations as in Chapter I. We see, then, that we can start with the present case and, as a corollary, derive the equations in Chapter I.

The second remark is, that when we use the present approach to compute the current densities on ϖ, then we can use the integral representations to compute the fields in the coaxial line only in the region enclosed by ϖ. Differently stated, we are not able to compute the fields below depth –d. In I, we can compute the fields at any depth inside the line. Perhaps, this is the trade-off between the two methods. We are confident that the present approach is correct and that it is the preferred one computationally since what we are striving for is to get the fields correctly at the opening, σ, of the coaxial line.

6. FAR FIELD

We will present two ways to compute the far field. First, we point out that we can express the far field in terms of an integral over σ (and an integral over S). The integral over σ, however, contains the magnetic current density on σ, a quantity we have not computed above. We are thus led to consider other representations of the far field. These representations will be for the magnetic field. To obtain representations for the electric field, we recall the following.

In the far field, the magnetic field can be expressed in the form

$$\mathbf{H}^r(\mathbf{r}) \sim \frac{\mathrm{e}^{-ikr}}{r} f(\hat{r})\hat{\alpha}(\hat{r}), \quad r \to \infty. \tag{6.1}$$

From Maxwell's equations

$$\mathbf{E}^r(\mathbf{r}) = \frac{1}{ikY}\nabla\times\mathbf{H}^r(\mathbf{r}) \sim \left\{\nabla\left[\frac{\mathrm{e}^{-ikr}}{r} f(\hat{r})\right]\times\hat{\alpha}(\hat{r}) + \frac{\mathrm{e}^{-ikr}}{r} f(\hat{r})\nabla\times\hat{\alpha}(\hat{r})\right\}, \quad r \to \infty. \tag{6.2}$$

But

$$\nabla\left[\frac{\mathrm{e}^{-ikr}}{r} f(\hat{r})\right]\times\hat{\alpha}(\hat{r}) \sim -ik\frac{\mathrm{e}^{-ikr}}{r} f(\hat{r})\hat{r}\times\hat{\alpha}(\hat{r}) + O\left(r^{-2}\right), \quad r \to \infty \tag{6.3}$$

and

$$\nabla\times\hat{\alpha}(\hat{r}) = O\left(r^{-1}\right), \quad r \to \infty. \tag{6.4}$$

Hence

$$\mathbf{E}^r(\mathbf{r}) \sim -Z\frac{\mathrm{e}^{-ikr}}{r} f(\hat{r})\hat{r}\times\hat{\alpha}(\hat{r}), \quad r \to \infty \tag{6.5}$$

or

$$\mathbf{E}^r(\mathbf{r}) = -Z\hat{r}\times\mathbf{H}^r(\mathbf{r}), \quad r \to \infty. \tag{6.6}$$

We turn to the derivation of the far-field representations of the magnetic field. The first comes from (2.3). We observe that

$$g(\mathbf{r},\mathbf{r}') \sim -\frac{\mathrm{e}^{-ikr'}\mathrm{e}^{ik\hat{r}'\cdot\mathbf{r}}}{4\pi r'}, \quad r' \to \infty \tag{6.7}$$

$$\nabla g(\mathbf{r},\mathbf{r}') \sim -ikg(\mathbf{r},\mathbf{r}')\hat{r}' , \quad \nabla\nabla g(\mathbf{r},\mathbf{r}') \sim -k^2 g(\mathbf{r},\mathbf{r}')\hat{r}'\hat{r}' , \quad r' \to \infty \tag{6.8}$$

and substitute in (2.3)

$$\begin{aligned} ikZ\mathbf{H}^r\left(\mathbf{r}'\right) &\sim -2\int_\tau \mathbf{M}_\tau(\mathbf{r})\cdot\left[-k^2 g(\mathbf{r},\mathbf{r}')\hat{r}'\hat{r}' + k^2 g(\mathbf{r},\mathbf{r}')\mathbf{I}\right]dS \\ &+2k^2 Z\hat{r}'\times\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS + \int_\sigma \mathbf{J}_\sigma(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS + \int_\tau \mathbf{J}_\tau(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS\right\} \\ &-k^2 Z\int_S\left\{\mathbf{J}_S(\mathbf{r})\times\hat{r}'g(\mathbf{r},\mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r})\times\hat{r}_i' g(\mathbf{r},\mathbf{r}_i')\right]_i\right\}dS , \quad r'\to\infty . \end{aligned} \tag{6.9}$$

We observe that

$$\mathbf{M}_\tau(\mathbf{r})\cdot\left[-\hat{r}'\hat{r}'+\mathbf{I}\right] = -\mathbf{M}_\tau(\mathbf{r})\cdot\left[\hat{r}'\times\left(\hat{r}'\times\mathbf{I}\right)\right] = \left[\hat{r}'\times\mathbf{M}_\tau(\mathbf{r})\right]\cdot\left(\hat{r}'\times\mathbf{I}\right) = -\hat{r}'\times\left[\hat{r}'\times\mathbf{M}_\tau(\mathbf{r})\right] \tag{6.10}$$

and that

$$\begin{aligned} \left[\mathbf{J}_S(\mathbf{r})\times\hat{r}_i'\right]_i &= \hat{x}\left[\left(\hat{y}\cdot\mathbf{J}_S(\mathbf{r})\right)\left(-\hat{z}\cdot\hat{r}'\right) - \left(\hat{z}\cdot\mathbf{J}_S(\mathbf{r})\right)\left(\hat{y}\cdot\hat{r}'\right)\right] \\ &\quad + \hat{y}\left[\left(\hat{z}\cdot\mathbf{J}_S(\mathbf{r})\right)\left(\hat{x}\cdot\hat{r}'\right) - \left(\hat{x}\cdot\mathbf{J}_S(\mathbf{r})\right)\left(-\hat{z}\cdot\hat{r}'\right)\right] - \hat{z}\left[\left(\hat{x}\cdot\mathbf{J}_S(\mathbf{r})\right)\left(\hat{y}\cdot\hat{r}'\right) - \left(\hat{y}\cdot\mathbf{J}_S(\mathbf{r})\right)\left(\hat{x}\cdot\hat{r}'\right)\right] \\ &= \hat{x}\left[\left(\hat{y}\cdot\hat{r}'\right)\left(-\hat{z}\cdot\mathbf{J}_S(\mathbf{r})\right) - \left(\hat{z}\cdot\hat{r}'\right)\left(\hat{y}\cdot\mathbf{J}_S(\mathbf{r})\right)\right] + \hat{y}\left[\left(\hat{z}\cdot\hat{r}'\right)\left(\hat{x}\cdot\mathbf{J}_S(\mathbf{r})\right) - \left(\hat{x}\cdot\hat{r}'\right)\left(-\hat{z}\cdot\mathbf{J}_S(\mathbf{r})\right)\right] \\ &\quad + \hat{z}\left[\left(\hat{x}\cdot\hat{r}'\right)\left(\hat{y}\cdot\mathbf{J}_S(\mathbf{r})\right) - \left(\hat{y}\cdot\hat{r}'\right)\left(\hat{x}\cdot\mathbf{J}_S(\mathbf{r})\right)\right] = \hat{r}'\times\left[\mathbf{J}_S(\mathbf{r})\right]_i . \end{aligned} \tag{6.11}$$

Substitution of the last two results in (6.9) gives

$$\begin{aligned} \mathbf{H}^r\left(\mathbf{r}'\right) &\sim -i2kY\hat{r}'\times\left[\hat{r}'\times\int_\tau \mathbf{M}_\tau(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS\right] \\ &- i2k\hat{r}'\times\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS + \int_\sigma \mathbf{J}_\sigma(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS + \int_\tau \mathbf{J}_\tau(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS\right\} \\ &- ik\hat{r}'\times\int_S\left\{\mathbf{J}_S(\mathbf{r})g(\mathbf{r},\mathbf{r}') - \left[\mathbf{J}_S(\mathbf{r})\right]_i g(\mathbf{r},\mathbf{r}_i')\right\}dS , \quad r'\to\infty \end{aligned} \tag{6.12}$$

where g is given by (6.7).

A second representation of the far field can be obtained from (I.6.1) with $\mathbf{\Gamma}_2$ replaced by $\mathbf{\Gamma}_1$

$$\int_D \left\{ \mathbf{E}^t(\mathbf{r}) \cdot \left[\nabla \times \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right] - \left[\nabla \times \nabla \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dV$$
$$= \int_\varpi \left\{ -\mathbf{E}^t(\mathbf{r}) \cdot \left[\hat{n} \times \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right] + \left[\hat{n} \times \nabla \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dS \,, \quad \mathbf{r}' \in D_f \,. \tag{6.13}$$

The region D_f is the upper-half space, excluding the monopole structure and the reflection of the region D about the *xy*-plane. We can think of it as representing the far-field region. Thus, the contribution of the volume integral is zero while, for the surface integral, we have

$$\int_\varpi \left\{ -\mathbf{E}^t(\mathbf{r}) \cdot \left[\hat{n} \times \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right] + \left[\hat{n} \times \nabla \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dS$$
$$= -\int_\sigma \left\{ \left[\hat{z} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dS$$
$$+ \int_\tau \left\{ \left[\hat{n} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') - ikZ \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dS$$
$$- ikZ \int_{S_a} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') dS - ikZ \int_{S_b} \left[\hat{n} \times \mathbf{H}^t(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') dS \tag{6.14}$$

or

$$-\int_\sigma \left\{ \left[\hat{z} \times \mathbf{E}^t(\mathbf{r}) \right] \cdot \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dS - \int_\tau \left\{ \mathbf{M}_t(\mathbf{r}) \cdot \nabla \times \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') + ikZ \mathbf{J}_\tau(\mathbf{r}) \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') \right\} dS$$
$$- ikZ \int_{S_a} \mathbf{J}_{S_a}(\mathbf{r}) \cdot \Gamma_1(\mathbf{r}, \mathbf{r}') dS - ikZ \int_{S_b} \mathbf{J}_{S_b}(\mathbf{r}) \cdot \Gamma_1(\mathbf{r}, \mathbf{r}') dS = 0 \,, \quad \mathbf{r}' \in D_f \,. \tag{6.15}$$

We combine this with (I.2.12) to get

$$-ikZ \int_S \mathbf{J}_S(\mathbf{r}) \cdot \Gamma_1(\mathbf{r}, \mathbf{r}') dS - \int_\tau \left\{ \mathbf{M}_t(\mathbf{r}) \cdot \nabla \times \Gamma_1(\mathbf{r}, \mathbf{r}') + ikZ \mathbf{J}_\tau(\mathbf{r}) \cdot \Gamma_1(\mathbf{r}, \mathbf{r}') \right\} dS$$
$$- ikZ \int_{S_a} \mathbf{J}_{S_a}(\mathbf{r}) \cdot \Gamma_1(\mathbf{r}, \mathbf{r}') dS - ikZ \int_{S_b} \mathbf{J}_{S_b}(\mathbf{r}) \cdot \boldsymbol{\Gamma}_1(\mathbf{r}, \mathbf{r}') dS = k^2 Z \mathbf{H}^r(\mathbf{r}') \,, \quad \mathbf{r}' \in D_f \tag{6.16}$$

or, in terms of the scalar Green's function (see (I.2.2)

$$\mathbf{H}^r(\mathbf{r}') = -\int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS$$
$$- \int_{S_a} \left\{ \mathbf{J}_{S_a}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') + \left[\mathbf{J}_{S_a}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS$$
$$- \int_{S_b} \left\{ \mathbf{J}_{S_b}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') + \left[\mathbf{J}_{S_b}(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS$$
$$- \int_\tau \left\{ \mathbf{J}_\tau(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') + \left[\mathbf{J}_\tau(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS$$

$$-\frac{1}{ikZ}\int_{\tau}\left\{\mathbf{M}_{\tau}(\mathbf{r})\cdot\nabla\left[\nabla g(\mathbf{r},\mathbf{r}')+\left(\nabla g(\mathbf{r},\mathbf{r}_i')\right)_i\right]+k^2\left[\mathbf{M}_{\tau}(\mathbf{r})g(\mathbf{r},\mathbf{r}')+\left(\mathbf{M}_{\tau}(\mathbf{r})\right)_i g(\mathbf{r},\mathbf{r}_i')\right]\right\}dS\,,$$

$$\mathbf{r}'\in D_f\,. \tag{6.17}$$

We proceed to evaluate this expression in the far field using (6.8)

$$\mathbf{H}^r\left(\mathbf{r}'\right)\sim -ik\hat{r}'\times\int_S\left\{\mathbf{J}_S(\mathbf{r})g(\mathbf{r},\mathbf{r}')+\left[\mathbf{J}_S(\mathbf{r})\right]_i g(\mathbf{r},\mathbf{r}_i')\right\}dS$$
$$-ik\hat{r}'\times\int_{S_a}\left\{\mathbf{J}_{S_a}(\mathbf{r})g(\mathbf{r},\mathbf{r}')+\left[\mathbf{J}_{S_a}(\mathbf{r})\right]_i g(\mathbf{r},\mathbf{r}_i')\right\}dS-ik\hat{r}'\times\int_{S_b}\left\{\mathbf{J}_{S_b}(\mathbf{r})g(\mathbf{r},\mathbf{r}')+\left[\mathbf{J}_{S_b}(\mathbf{r})\right]_i g(\mathbf{r},\mathbf{r}_i')\right\}dS$$
$$-ik\hat{r}'\times\int_{\tau}\mathbf{J}_{\tau}(\mathbf{r})\left[g(\mathbf{r},\mathbf{r}')+g(\mathbf{r},\mathbf{r}_i')\right]dS-ikY\hat{r}'\times\left\{\hat{r}'\times\int_{\tau}\mathbf{M}_{\tau}(\mathbf{r})\left[g(\mathbf{r},\mathbf{r}')+g(\mathbf{r},\mathbf{r}_i')\right]dS\right\},\quad r'\to\infty \tag{6.18}$$

where g is given by (6.7). To obtain this result, we used (6.10) and (6.11). We also used the fact that, on τ, the electric and magnetic current densities are transverse to the z-axis. As a consequence

$$\left[\mathbf{J}_{\tau}\right]_i=\mathbf{J}_{\tau}\,,\quad\left[\mathbf{M}_{\tau}\right]_i=\mathbf{M}_{\tau}\,,\quad\mathbf{M}_{\tau}\cdot\hat{r}_i'=\mathbf{M}_{\tau}\cdot\hat{r}_i\,. \tag{6.19}$$

Finally,

$$\nabla\left[\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\sim\nabla\left[-ikg(\mathbf{r},\mathbf{r}_i')\hat{r}_i'\right]_i=-ik\nabla\left[g(\mathbf{r},\mathbf{r}_i')\hat{r}'\right]\sim -k^2g(\mathbf{r},\mathbf{r}_i')\hat{r}_i'\hat{r}'\,,\quad r'\to\infty\,. \tag{6.20}$$

The far field expression (6.18) differs substantially from the one in (6.12). It may even be argued that (6.12) is simpler to compute. Equation (6.18) demonstrates, however, that, when it comes to the far field, we have an image theory in place. We proceed to show this. Define a current density $\mathbf{J}_{S_i}$ on the image S_i of S (about the xy-plane) as follows

$$\left.\begin{array}{l}\hat{z}\times\mathbf{J}_{S_i}(\mathbf{r}_i)=-\hat{z}\times\mathbf{J}_S(\mathbf{r})\,,\quad\mathbf{r}\in S\\ \hat{z}\cdot\mathbf{J}_{S_i}(\mathbf{r}_i)=\hat{z}\cdot\mathbf{J}_S(\mathbf{r})\,,\quad\mathbf{r}\in S\end{array}\right\}. \tag{6.21}$$

This statement is in agreement with how electric currents are imaged in the presnce of a perfectly conducting half-space with normal along the z-axis. We make similar definitions on the images of the rest of the surfaces in (6.18). For the magnetic current density, we define its image on τ_i as follows

$$\hat{z}\times\mathbf{M}_{\tau_i}(\mathbf{r}_i)=\hat{z}\times\mathbf{M}_{\tau}(\mathbf{r})\,,\quad\mathbf{r}\in\tau\,. \tag{6.22}$$

We recall from (6.19) that the magnetic current density does not have a component along the z-axis.

We examine the second term in the first integral in (6.18). By (6.7)

$$\int_S \left[\mathbf{J}_S(\mathbf{r})\right]_i g(\mathbf{r},\mathbf{r}_i')dS = \frac{-e^{-ikr'}}{4\pi r'}\int_S \left[\mathbf{J}_S(\mathbf{r})\right]_i e^{ik\hat{r}_i'\cdot\mathbf{r}}\, dS\,. \tag{6.23}$$

Using (6.21) and the fact that

$$\hat{r}_i'\cdot\mathbf{r} = \hat{r}'\cdot\mathbf{r}_i \tag{6.24}$$

we get that

$$\int_S \left[\mathbf{J}_S(\mathbf{r})\right]_i g(\mathbf{r},\mathbf{r}_i')dS = \frac{e^{-ikr'}}{4\pi r'}\int_{S_i} \mathbf{J}_{S_i}(\mathbf{r}_i)\, e^{ik\hat{r}'\cdot\mathbf{r}_i}\, dS_i\,. \tag{6.25}$$

This is an integral defined over the image of the surface S. It is worth writing the whole expression (6.18) out

$$\begin{aligned}
&\mathbf{H}^r(\mathbf{r}') \sim -ik\left(\frac{-e^{-ikr'}}{4\pi r'}\right)\hat{r}'\times\left\{\int_S \mathbf{J}_S(\mathbf{r})\,e^{ik\hat{r}'\cdot\mathbf{r}}\,dS - \int_{S_i}\mathbf{J}_{S_i}(\mathbf{r}_i)\,e^{ik\hat{r}'\cdot\mathbf{r}_i}\,dS_i\right\}\\
&-ik\left(\frac{-e^{-ikr'}}{4\pi r'}\right)\hat{r}'\times\left\{\int_{S_a}\mathbf{J}_{S_a}(\mathbf{r})\,e^{ik\hat{r}'\cdot\mathbf{r}}\,dS - \int_{(S_a)_i}\mathbf{J}_{(S_a)_i}(\mathbf{r}_i)\,e^{ik\hat{r}'\cdot\mathbf{r}_i}\,dS_i\right\}\\
&-ik\left(\frac{-e^{-ikr'}}{4\pi r'}\right)\hat{r}'\times\left\{\int_{S_b}\mathbf{J}_{S_b}(\mathbf{r})\,e^{ik\hat{r}'\cdot\mathbf{r}}\,dS - \int_{(S_b)_i}\mathbf{J}_{(S_b)_i}(\mathbf{r}_i)\,e^{ik\hat{r}'\cdot\mathbf{r}_i}\,dS_i\right\}\\
&-ik\left(\frac{-e^{-ikr'}}{4\pi r'}\right)\hat{r}'\times\left\{\int_{\tau}\mathbf{J}_{\tau}(\mathbf{r})\,e^{ik\hat{r}'\cdot\mathbf{r}}\,dS - \int_{\tau_i}\mathbf{J}_{\tau_i}(\mathbf{r}_i)\,e^{ik\hat{r}'\cdot\mathbf{r}_i}\,dS_i\right\}\\
&-ikY\left(\frac{-e^{-ikr'}}{4\pi r'}\right)\hat{r}'\times\left\{\hat{r}'\times\left[\int_{\tau}\mathbf{M}_{\tau}(\mathbf{r})\,e^{ik\hat{r}'\cdot\mathbf{r}}\,dS + \int_{\tau_i}\mathbf{M}_{\tau_i}(\mathbf{r}_i)\,e^{ik\hat{r}'\cdot\mathbf{r}_i}\,dS_i\right]\right\},\quad r'\to\infty\,.
\end{aligned} \tag{6.26}$$

Thus, we have an expression for the far field in terms of integrals over the surfaces S, S_a, S_b, τ and their images about the xy-plane. We note that there is no integral over σ.

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CHAPTER 3
MONOPOLE AS AN EXTENSION OF THE CENTER CONDUCTOR

1. INTRODUCTION

In this chapter, we use results from Chapters 1 and 2 to develop a system of integral equations for geometries as in Figures 1.1 and 1.2. In these figures, the monopole is an extension of the center conductor of the coaxial transmission line. The diameter of the monopole in Figure 1.2 is not restricted in any way. It can be smaller than that of the center conductor or even greater than that of the outer conductor. Also, the height h of the monopole above ground is variable and it is allowed to become zero. Figure 1.1 is a special case of Figure 1.2. We proceed to derive integral equations for the geometry of Figure 1.2. Equation numbers from Chapters I and II are preceded by Roman numerals I and II, respectively.

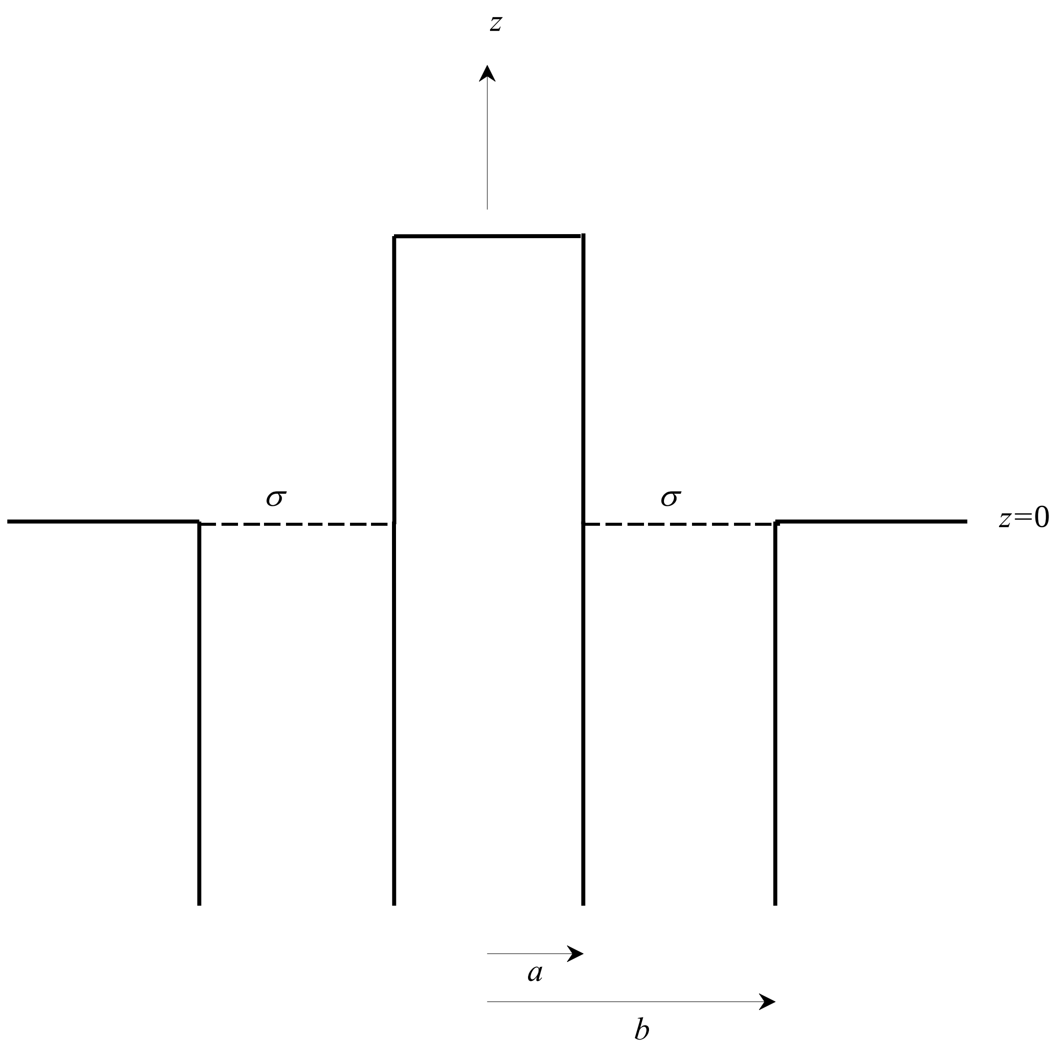

Figure 1.1. Monopole as extension of center conductor of coaxial line

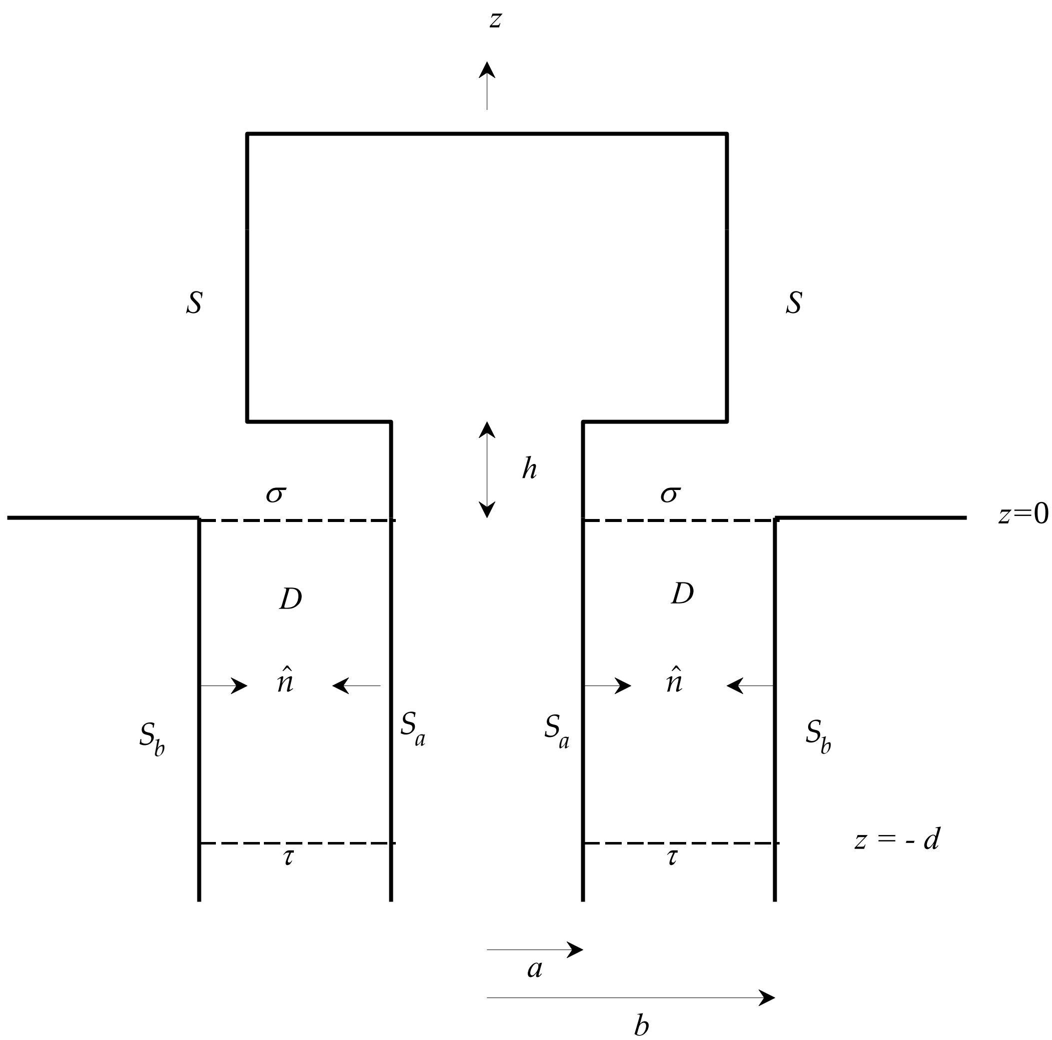

Figure 1.2. A more general monopole geometry

For $h > 0$, we denote the surface of the structure in the upper-half space ($z > 0$) by *S*. If $h = 0$, then we consider the part that lies on the *xy*-plane as part of the inner conductor S_a. We first consider the case $h > 0$.

2. CENTER CONDUCTOR EXTENDING ABOVE THE PLANE ($h > 0$)

The unknowns are given in (I.4.6)-(I.4.9). The currents $\mathbf{J}_S$ and $\mathbf{J}_{S_a}$ must be equal on the *xy*-plane. The integral equations are given by (I.5.2), (I.5.6), (I.7.1), and (I.7.2).

If the coaxial line is to be terminated, as in Figure 2, then we have two additional unknowns and they are given by (II.2.1). The integral equations are given by (II.3.1), (II.3.2), and (II.4.1) through (II.4.4).

3. CENTER CONDUCTOR EXTENDING TO THE PLANE ($h = 0$)

The unknowns and integral equations are as in the previous section.

4. REMARKS

We must make sure that at the point $\mathbf{r}' = (a, \varphi', 0)$, integral equations (I.5.2) and (I.7.1) give the same answer. These equations are

$$-2\hat{n}' \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS \right\}$$
$$- \hat{n}' \times \int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS = \frac{1}{2} \mathbf{J}_S(\mathbf{r}') \quad , \quad \mathbf{r}' \in S \, . \tag{4.1}$$

and

$$-\hat{n}' \times \int_{S_a} \left\{ \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') - \left[\mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS$$
$$- \hat{n}' \times \int_{S_b} \left\{ \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') - \left[\mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS$$
$$- 2\hat{n}' \times \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS = \frac{1}{2} \mathbf{J}_a(\mathbf{r}') \, , \quad \mathbf{r}' \in S_a \, . \tag{4.2}$$

If $\mathbf{r}' = (a, \varphi', 0)$, these equations become

$$-2\hat{\rho}' \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS \right\}$$
$$- \hat{\rho}' \times \int_S \left\{ \mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') + \left[\mathbf{J}_S(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') \right]_i \right\} dS = \frac{1}{2} \mathbf{J}_S(\mathbf{r}') \, , \quad \mathbf{r}' = (a, \varphi', 0) \tag{4.3}$$

and

$$-\hat{\rho}' \times \int_{S_a} \left\{ \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') - \left[\mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') \right]_i \right\} dS$$
$$- \hat{\rho}' \times \int_{S_b} \left\{ \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') - \left[\mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') \right]_i \right\} dS$$
$$- 2\hat{\rho}' \times \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS = \frac{1}{2} \mathbf{J}_a(\mathbf{r}') \, , \quad \mathbf{r}' = (a, \varphi', 0) \, . \tag{4.4}$$

The $\hat{\rho}'$-component of both equations is zero. From (4.3)

$$\frac{1}{2} \hat{\varphi}' \cdot \mathbf{J}_S(\mathbf{r}') = 2\hat{z} \cdot \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS \right\} \tag{4.5}$$

while from (4.4)

$$\frac{1}{2}\hat{\varphi}'\cdot\mathbf{J}_a(\mathbf{r}') = 2\hat{z}\cdot\int_\sigma \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + 2\hat{z}\cdot\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}. \tag{4.6}$$

From these two

$$\hat{\varphi}'\cdot\mathbf{J}_S\left(\mathbf{r}'\right) = \hat{\varphi}'\cdot\mathbf{J}_a(\mathbf{r}'), \quad \mathbf{r}' = (a,\varphi',0). \tag{4.7}$$

For the z-component, we get from (4.3) that

$$\begin{aligned}\frac{1}{2}\hat{z}\cdot\mathbf{J}_S\left(\mathbf{r}'\right) &= -2\hat{\varphi}'\cdot\int_S \mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS \\ &\quad -2\hat{\varphi}'\cdot\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_\sigma \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}\end{aligned} \tag{4.8}$$

while from (4.4)

$$\begin{aligned}\frac{1}{2}\hat{z}\cdot\mathbf{J}_a\left(\mathbf{r}'\right) &= -2\hat{\varphi}'\cdot\int_\sigma \mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS = -2\int_\sigma \mathbf{J}_\sigma(\mathbf{r})\cdot\left[\hat{\varphi}'\times\nabla' g(\mathbf{r},\mathbf{r}')\right]dS \\ &= -2\int_\sigma \mathbf{J}_\sigma(\mathbf{r})\cdot\left[-\hat{z}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial\rho'} + \hat{\rho}'\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z'}\right]dS = 0.\end{aligned} \tag{4.9}$$

With this last result, we can write for (4.8)

$$\hat{z}\cdot\mathbf{J}_S\left(\mathbf{r}'\right) = -4\hat{\varphi}'\cdot\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_S \mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}. \tag{4.10}$$

From (I.8.4)

$$\hat{\rho}'\cdot\mathbf{J}_\sigma(\mathbf{r}') = -\hat{\varphi}'\cdot\left\{\int_{S_a}\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_{S_b}\mathbf{J}_b(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS + \int_S \mathbf{J}_S(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\right\}, \quad \mathbf{r}' = (a,\varphi',0). \tag{4.11}$$

From the last two

$$\hat{z}\cdot\mathbf{J}_S\left(\mathbf{r}'\right) = 4\hat{\rho}'\cdot\mathbf{J}_\sigma(\mathbf{r}'), \quad \mathbf{r}' = (a,\varphi',0). \tag{4.12}$$

But $\hat{\rho}' \cdot \mathbf{J}_\sigma(\mathbf{r}')$ is the normal component of an electric current density on a perfect conductor and, hence, it must be zero. We conclude then that

$$\hat{z} \cdot \mathbf{J}_S\left(\mathbf{r}'\right) = 0 \,, \quad \mathbf{r}' = (a, \varphi', 0) \,. \tag{4.13}$$

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PART 2
INTEGRAL EQUATIONS FOR THE FIELDS OF A COAXIAL LINE RADIATING INTO A HALF-SPACE

ABSTRACT

This is the second Part in a report on the formulation of the problem of radiation of a coaxial line into a half-space in terms of BIEs. In it, we use the results of Part 1 to derive BIEs for the fields of a coaxial line radiating into an otherwise empty half-space. The relationship of wavelength to the radii of the line is such that the input wave is a TEM wave. We take advantage of the circular symmetry of the line to reduce the vector integral equations to three scalar equations. The unknown electric current densities on the walls of the line are expressed as infinite series in the natural modes of the line, the coefficients of the modes being the unknowns. We point out that any of the three integral equations can be solved numerically using the method of weighted residuals. This method results in an infinite system of linear equations with an infinite number of unknowns. The actual solution of a truncated version of such a system will be given in Part 3 of this report. We conclude by deriving expressions for the far fields in the half-space in which the coaxial line radiates.

1. INTRODUCTION

We use here the integral equations developed in Part 1 to determine the fields of the structure in Figure 1.1: a coaxial transmission line, terminating in a ground plane and radiating over the upper-half space. All surfaces are perfectly conducting. The ground plane is infinite and the coaxial line extends to infinity in the lower-half space. The inner and outer radii of the transmission line are *a* and *b*, respectively. We assume that the line supports only a TEM wave. For this, it is sufficient that (references 1 and 2)

$$k\frac{a+b}{2}<1 \tag{1.1}$$

where k is the wavenumber of the time-harmonic ($\mathrm{e}^{+i\omega t}$) electromagnetic wave in the line.

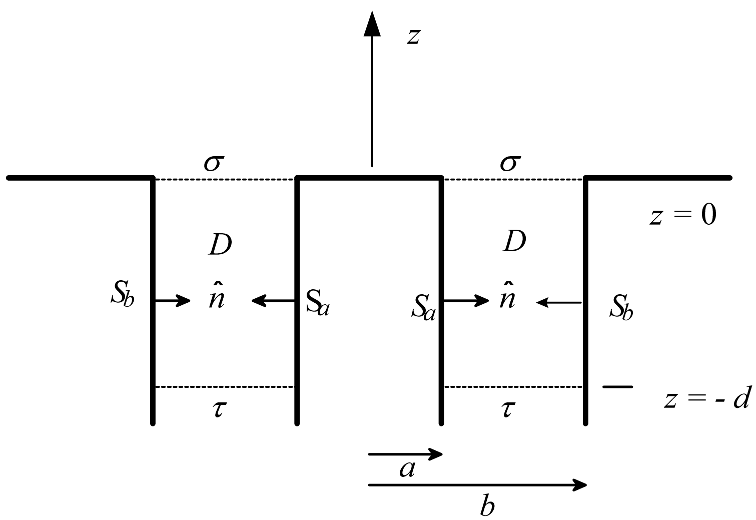

Figure 1.1. The semi-infinite coaxial line of inner radius *a* and outer radius *b* is fed at $z=-\infty$. At $z=0$, it opens up into a half-space, its outer conductor becoming a plane that extends to infinity. All surfaces are perfectly conducting.

The excitation of the line occurs at $z=-\infty$ and results in a TEM wave with fields

$$\mathbf{E}^g(\mathbf{r})=\frac{V\,\mathrm{e}^{-ikz}}{\ln\left(\dfrac{b}{a}\right)\rho}\hat{\rho}\,,\quad \mathbf{H}^g(\mathbf{r})=\frac{YV\,\mathrm{e}^{-ikz}}{\ln\left(\dfrac{b}{a}\right)\rho}\hat{\varphi}\,. \tag{1.2}$$

where V is the voltage of the inner conductor with respect to the outer, and Y is the free-space admittance. We use cylindrical coordinates (ρ,φ,z). Since the line is not infinite, we have also an induced wave in the line with fields {$\mathbf{E}^i,\mathbf{H}^i$}. The total fields {$\mathbf{E}^t,\mathbf{H}^t$} in the coaxial line are the sum of the generator and induced fields

$$\mathbf{E}^t=\mathbf{E}^g+\mathbf{E}^i\,,\quad \mathbf{H}^t=\mathbf{H}^g+\mathbf{H}^i\,. \tag{1.3}$$

In the upper-half space we have radiated fields $\{\mathbf{E}^r, \mathbf{H}^r\}$. We proceed to determine both sets of fields by first deriving integral equations and then solving them numerically.

2. FIELDS INDUCED BY COAXIAL LINE

Since the generator fields are independent of the angular coordinate, and since the geometry has circular symmetry, we expect the induced fields to be independent of φ. Thus, according to Appendix A, we can have a TEM wave traveling down the line, given by

$$\mathbf{E}^i(\mathbf{r}) = \frac{V\,\mathrm{e}^{+ikz}}{\ln\left(\frac{b}{a}\right)\rho}\,\hat{\rho}\,, \quad \mathbf{H}^i(\mathbf{r}) = -\frac{YV\,\mathrm{e}^{+ikz}}{\ln\left(\frac{b}{a}\right)\rho}\,\hat{\varphi}\,. \tag{2.1}$$

In addition, we can have the zeroth-order mode of a TM wave, given by (A3.23), (A3.24), and (A3.27) in Appendix A

$${}^{TM}E_{0ne}^{z-}(\rho,\varphi,z) = {\nu_{0n}}^2\,\mathrm{e}^{i\lambda_{0n}z}\,P_{0n}(\rho) \tag{2.2}$$

$${}^{TM}E_{0ne}^{\rho-}(\rho,\varphi,z) = i\lambda_{0n}\,\mathrm{e}^{i\lambda_{0n}z}\,{P_{0n}}'(\rho) \tag{2.3}$$

$${}^{TM}H_{0ne}^{\varphi-}(\rho,\varphi,z) = ikY\,\mathrm{e}^{i\lambda_{0n}z}\,{P_{0n}}'(\rho) \tag{2.4}$$

with ν_{0n} a root of the equation

$$J_0(\nu_{0n}a)Y_0(\nu_{0n}b) - J_0(\nu_{0n}b)Y_0(\nu_{0n}a) = 0\,, \quad n = 1,2,\ldots \tag{2.5}$$

and

$$\lambda_{0n} = \begin{cases} \sqrt{k^2 - {\nu_{0n}}^2}\,, & k^2 > {\nu_{0n}}^2 \\ -i\sqrt{{\nu_{0n}}^2 - k^2}\,, & {\nu_{0n}}^2 > k^2 \end{cases} \tag{2.6}$$

while

$$P_{0n}(\rho) = J_0(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_0(\nu_{0n}\rho)\,. \tag{2.7}$$

Similarly, we can have the zeroth-order mode of the TE wave (see Appendix A)

$${}^{TE}H_{0ne}^{z-}(\rho,\varphi,z) = {\mu_{0n}}^2\,\mathrm{e}^{i\kappa_{0n}z}\,R_{0n}(\rho) \tag{2.8}$$

$${}^{TE}H_{0ne}^{\rho-}(\rho,\varphi,z) = i\kappa_{0n}\,\mathrm{e}^{i\kappa_{0n}z}\,{R_{0n}}'(\rho) \tag{2.9}$$

$${}^{TE}E_{0ne}^{\varphi-}(\rho,\varphi,z) = ikZ\,\mathrm{e}^{i\kappa_{0n}z}\,{R_{0n}}'(\rho) \tag{2.10}$$

with μ_{0n} a root of the equation

$$J_0'(\mu_{0n}a)Y_0'(\mu_{0n}b) - J_0'(\mu_{0n}b)Y_0'(\mu_{0n}a) = 0\,, \quad n = 1, 2, \ldots \tag{2.11}$$

and

$$\kappa_{0n} = \begin{cases} \sqrt{k^2 - \mu_{0n}{}^2}\,, & k^2 > \mu_{0n}{}^2 \\ -i\sqrt{\mu_{0n}{}^2 - k^2}\,, & \mu_{0n}{}^2 > k^2 \end{cases} \tag{2.12}$$

while

$$R_{0n}(\rho) = J_0(\mu_{0n}\rho) - \frac{J_0'(\mu_{0n}a)}{Y_0'(\mu_{0n}a)} Y_0(\mu_{0n}\rho)\,. \tag{2.13}$$

The first few roots of (2.5) and (2.11) are given in (reference 3) for selected values of $\chi = b/a$. Also, Mathematica® (reference 4) has special subroutines that compute the roots of these equations.

From the above, we can write that the magnetic field induced in the coaxial line is

$$\mathbf{H}^i(\mathbf{r}) = -A\frac{YV\,\mathrm{e}^{+ikz}}{\ln\left(\frac{b}{a}\right)\rho}\hat{\varphi} + V\sum_{n=1}^{\infty}\left\{B_n ikY\,\mathrm{e}^{i\lambda_{0n}z}\,P_{0n}'(\rho)\hat{\varphi} + C_n\,\mathrm{e}^{i\kappa_{0n}z}\left[i\kappa_{0n}R_{0n}'(\rho)\hat{\rho} + \mu_{0n}{}^2R_{0n}(\rho)\hat{z}\right]\right\} \tag{2.14}$$

where A, B_n, and C_n are unknown constants. The constant A is dimensionless, B_n has dimension of length, and C_n has dimension of length-mhos.

3. INTEGRAL EQUATIONS AND ELECTRIC CURRENT DENSITIES

From (5.6), (7.1), and (7.2) of Chapter 1 of Part 1, the integral equations for the present problem are

$$\hat{z} \times \left\{ \int_{S_a} \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS + \int_{S_b} \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS \right\} = \mathbf{J}_\sigma(\mathbf{r}') \, , \; \mathbf{r}' \in \sigma \tag{3.1}$$

and

$$\begin{aligned} &-\hat{n}' \times \int_{S_a} \left\{ \mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') - \left[\mathbf{J}_a(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS \\ &\qquad -\hat{n}' \times \int_{S_b} \left\{ \mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') - \left[\mathbf{J}_b(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}_i') \right]_i \right\} dS \\ &\qquad -2\hat{n}' \times \int_{\sigma} \mathbf{J}_\sigma(\mathbf{r}) \times \nabla g(\mathbf{r}, \mathbf{r}') dS = \frac{1}{2} \begin{cases} \mathbf{J}_a(\mathbf{r}') \, , & \mathbf{r}' \in S_a \\ \mathbf{J}_b(\mathbf{r}') \, , & \mathbf{r}' \in S_b \end{cases} . \end{aligned} \tag{3.2}$$

The current densities in these equations are defined in (4.6), (4.7), and (4.8) of Chapter 1 of Part 1.

$$\mathbf{J}_a(\mathbf{r}) = \hat{n} \times \mathbf{H}^t(\mathbf{r}) \, , \; \mathbf{r} \in S_a \; , \quad \mathbf{J}_b(\mathbf{r}) = \hat{n} \times \mathbf{H}^t(\mathbf{r}) \, , \; \mathbf{r} \in S_b \; , \quad \mathbf{J}_\sigma(\mathbf{r}) = -\hat{z} \times \mathbf{H}^t(\mathbf{r}) \, , \; \mathbf{r} \in \sigma \tag{3.3}$$

where $\mathbf{H}^t$ is defined in (1.3).

In terms of the fields in (1.2) and (2.14), we have

$$\mathbf{J}_\sigma(\mathbf{r}) = (1 - A) \frac{YV}{\ln\left(\frac{b}{a}\right) \rho} \hat{\rho} + iV \sum_{n=1}^{\infty} \left\{ B_n k Y P_{0n}{}'(\rho) \hat{\rho} - C_n \kappa_{0n} R_{0n}{}'(\rho) \hat{\varphi} \right\} \tag{3.4}$$

while

$$\mathbf{J}_a(\mathbf{r}) = \frac{YV}{\ln\left(\frac{b}{a}\right) a} \left(\mathrm{e}^{-ikz} - A \, \mathrm{e}^{+ikz} \right) \hat{z} + V \sum_{n=1}^{\infty} \left\{ B_n ikY \, \mathrm{e}^{i\lambda_{0n} z} \; P_{0n}{}'(a) \hat{z} - C_n \, \mathrm{e}^{i\kappa_{0n} z} \; \mu_{0n}{}^2 R_{0n}(a) \hat{\varphi} \right\} \tag{3.5}$$

and

$$\mathbf{J}_b(\mathbf{r}) = -\frac{YV}{\ln\left(\frac{b}{a}\right) b} \left(\mathrm{e}^{-ikz} - A \, \mathrm{e}^{+ikz} \right) \hat{z} - V \sum_{n=1}^{\infty} \left\{ B_n ikY \, \mathrm{e}^{i\lambda_{0n} z} \; P_{0n}{}'(b) \hat{z} - C_n \, \mathrm{e}^{i\kappa_{0n} z} \; \mu_{0n}{}^2 R_{0n}(b) \hat{\varphi} \right\} \tag{3.6}$$

But from (2.7)

$$P_{0n}{}'(a) = \nu_{0n}\left[J_0{}'(\nu_{0n}a) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_0{}'(\nu_{0n}a)\right]$$
$$= \frac{\nu_{0n}}{Y_0(\nu_{0n}a)}\left[J_0{}'(\nu_{0n}a)Y_0(\nu_{0n}a) - J_0(\nu_{0n}a)Y_0{}'(\nu_{0n}a)\right] = -\frac{2}{\pi a Y_0(\nu_{0n}a)} \tag{3.7}$$

where, above, we made use of the Wronskian relationship for Bessel functions (reference 3). Similarly, from (2.13)

$$R_{0n}(a) = \frac{1}{Y_0{}'(\mu_{0n}a)}\left[J_0(\mu_{0n}a)Y_0{}'(\mu_{0n}a) - J_0{}'(\mu_{0n}a)Y_0(\mu_{0n}a)\right] = \frac{2}{\pi\mu_{0n}aY_0{}'(\mu_{0n}a)}\,. \tag{3.8}$$

With these two results, we can write for (3.5)

$$\mathbf{J}_a(\mathbf{r}) = \frac{YV}{a\ln(\chi)}\left(\mathrm{e}^{-ikz} - A\,\mathrm{e}^{+ikz}\right)\hat{z} - \frac{2V}{\pi a}\sum_{n=1}^{\infty}\left[\frac{ikYB_n}{Y_0(\nu_{0n}a)}\mathrm{e}^{i\lambda_{0n}z}\;\hat{z} - \frac{\mu_{0n}C_n}{Y_1(\mu_{0n}a)}\mathrm{e}^{i\kappa_{0n}z}\;\hat{\varphi}\right] \tag{3.9}$$

where $\chi = b/a$.

From (3.15) and (4.10) of (reference 1), we can write (2.7) and (2.13) in the form

$$P_{0n}(\rho) = J_0(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}b)}{Y_0(\nu_{0n}b)}Y_0(\nu_{0n}\rho)\,, \quad R_{0n}(\rho) = J_0(\mu_{0n}\rho) - \frac{J_0{}'(\mu_{0n}b)}{Y_0{}'(\mu_{0n}b)}Y_0(\mu_{0n}\rho)\,. \tag{3.10}$$

Proceeding as above, we find that

$$P_{0n}{}'(b) = -\frac{2}{\pi bY_0(\nu_{0n}b)}\,, \quad R_{0n}(b) = -\frac{2}{\pi\mu_{0n}bY_1(\mu_{0n}b)} \tag{3.11}$$

and

$$\mathbf{J}_b(\mathbf{r}) = -\frac{YV}{\ln(\chi)b}\left(\mathrm{e}^{-ikz} - A\,\mathrm{e}^{+ikz}\right)\hat{z} + \frac{2V}{\pi b}\sum_{n=1}^{\infty}\left\{\frac{ikYB_n}{Y_0(\nu_{0n}b)}\mathrm{e}^{i\lambda_{0n}z}\;\hat{z} - \frac{\mu_{0n}C_n}{Y_1(\mu_{0n}b)}\mathrm{e}^{i\kappa_{0n}z}\;\hat{\varphi}\right\}. \tag{3.12}$$

In the next section, we substitute the current densities in the integral equations and resolve the latter into their components.

4. SCALAR FORM OF INTEGRAL EQUATIONS

We substitute next expressions (3.4), (3.9), and (3.12) for the electric current densities in the integral equations (3.1) and (3.2). First, we write

$$\mathbf{J}_\sigma(\mathbf{r}) = u_\sigma(\mathbf{r})\hat{\rho} + v_\sigma(\mathbf{r})\hat{\varphi}\,, \quad \mathbf{J}_a(\mathbf{r}) = v_a(\mathbf{r})\hat{\varphi} + h_a(\mathbf{r})\hat{z}\,, \quad \mathbf{J}_b(\mathbf{r}) = v_b(\mathbf{r})\hat{\varphi} + h_b(\mathbf{r})\hat{z} \tag{4.1}$$

where

$$u_\sigma(\mathbf{r}) = (1-A)\frac{YV}{\ln(\chi)\rho} + ikYV\sum_{n=1}^{\infty} B_n P_{0n}{}'(\rho)\,, \quad v_\sigma(\mathbf{r}) = -iV\sum_{n=1}^{\infty} C_n \kappa_{0n} R_{0n}{}'(\rho) \tag{4.2}$$

$$v_a(\mathbf{r}) = \frac{2V}{\pi a}\sum_{n=1}^{\infty}\frac{\mu_{0n}C_n}{Y_1(\mu_{0n}a)}\mathrm{e}^{i\kappa_{0n}z}, \quad h_a(\mathbf{r}) = \frac{YV}{a\ln(\chi)}\left(\mathrm{e}^{-ikz} - A\,\mathrm{e}^{+ikz}\right) - \frac{2ikYV}{\pi a}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\mathrm{e}^{i\lambda_{0n}z} \tag{4.3}$$

$$v_b(\mathbf{r}) = -\frac{2V}{\pi b}\sum_{n=1}^{\infty}\frac{\mu_{0n}C_n}{Y_1(\mu_{0n}b)}\mathrm{e}^{i\kappa_{0n}z}, \quad h_b(\mathbf{r}) = -\frac{YV}{b\ln(\chi)}\left(\mathrm{e}^{-ikz} - A\,\mathrm{e}^{+ikz}\right) + \frac{2ikYV}{\pi b}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}b)}\mathrm{e}^{i\lambda_{0n}z}\,. \tag{4.4}$$

Compute

$$\begin{aligned}\hat{z}\times\left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')\right] &= \mathbf{J}_a(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} - \nabla g(\mathbf{r},\mathbf{r}')\hat{z}\cdot\mathbf{J}_a(\mathbf{r}) \\ &= \left[v_a(\mathbf{r})\hat{\varphi} + h_a(\mathbf{r})\hat{z}\right]\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} - \nabla g(\mathbf{r},\mathbf{r}')\hat{z}\cdot\left[v_a(\mathbf{r})\hat{\varphi} + h_a(\mathbf{r})\hat{z}\right] \\ &= v_a(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\hat{\varphi} - h_a(\mathbf{r})\nabla_t g(\mathbf{r},\mathbf{r}')\end{aligned} \tag{4.5}$$

where the subscript t stands for "transverse to the z-axis". A similar expression exists for the outer conductor. Substitute in (3.1)

$$\begin{aligned}&\int_{S_a}\left[v_a(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\hat{\varphi} - h_a(\mathbf{r})\nabla_t g(\mathbf{r},\mathbf{r}')\right]dS + \int_{S_b}\left[v_b(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\hat{\varphi} - h_b(\mathbf{r})\nabla_t g(\mathbf{r},\mathbf{r}')\right]dS \\ &\qquad = u_\sigma(\mathbf{r}')\hat{\rho}' + v_\sigma(\mathbf{r}')\hat{\varphi}'\,,\ \mathbf{r}'\in\sigma\,.\end{aligned} \tag{4.6}$$

This is the first integral equation in terms of scalar functions.

Compute

$$\begin{aligned}&\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\mathbf{J}_a(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i \\ &= \left[v_a(\mathbf{r})\hat{\varphi} + h_a(\mathbf{r})\hat{z}\right]\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\left(v_a(\mathbf{r})\hat{\varphi} + h_a(\mathbf{r})\hat{z}\right)\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i \\ &= v_a(\mathbf{r})\left\{\hat{\varphi}\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\hat{\varphi}\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]_i\right\} + h_a(\mathbf{r})\hat{z}\times\nabla\left(g(\mathbf{r},\mathbf{r}') - g(\mathbf{r},\mathbf{r}_i')\right)\end{aligned}$$

$$= v_a(\mathbf{r})\left\{\hat{\rho}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} - \hat{z}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho} - \hat{\rho}\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial z} - \hat{z}\frac{\partial g(\mathbf{r},\mathbf{r}_i')}{\partial \rho}\right\} + h_a(\mathbf{r})\hat{z}\times\nabla\left(g(\mathbf{r},\mathbf{r}') - g(\mathbf{r},\mathbf{r}_i')\right)$$

$$= v_a(\mathbf{r})\left[\hat{\rho}\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z} - \hat{z}\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}\right] + h_a(\mathbf{r})\hat{z}\times\nabla g_D(\mathbf{r},\mathbf{r}') \tag{4.7}$$

where

$$g_D(\mathbf{r},\mathbf{r}') = g(\mathbf{r},\mathbf{r}') - g(\mathbf{r},\mathbf{r}_i'), \quad g_N(\mathbf{r},\mathbf{r}') = g(\mathbf{r},\mathbf{r}') + g(\mathbf{r},\mathbf{r}_i') \tag{4.8}$$

the Dirichlet and Neumann Green's functions for the $z = 0$ plane. Equation (4.7) is also valid when the subscript a is replaced by b. We also compute

$$\mathbf{J}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') = \left[u_\sigma(\mathbf{r})\hat{\rho} + v_\sigma(\mathbf{r})\hat{\varphi}\right]\times\nabla g(\mathbf{r},\mathbf{r}')$$

$$= \left[\hat{\rho}v_\sigma(\mathbf{r}) - \hat{\varphi}u_\sigma(\mathbf{r})\right]\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} + \hat{z}\left[u_\sigma(\mathbf{r})\frac{1}{\rho}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \varphi} - v_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho}\right]. \tag{4.9}$$

Substitute these expressions in the first of (3.2) ($\mathbf{r}' \in S_a$, $\hat{n}' = \hat{\rho}'$)

$$-\hat{\rho}'\times\int_{S_a}\left\{v_a(\mathbf{r})\left[\hat{\rho}\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z} - \hat{z}\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}\right] + h_a(\mathbf{r})\hat{z}\times\nabla g_D(\mathbf{r},\mathbf{r}')\right\}dS$$

$$-\hat{\rho}'\times\int_{S_b}\left\{v_b(\mathbf{r})\left[\hat{\rho}\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z} - \hat{z}\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}\right] + h_b(\mathbf{r})\hat{z}\times\nabla g_D(\mathbf{r},\mathbf{r}')\right\}dS$$

$$-2\hat{\rho}'\times\int_{\sigma}\left\{\left[\hat{\rho}v_\sigma(\mathbf{r}) - \hat{\varphi}u_\sigma(\mathbf{r})\right]\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} + \hat{z}\left[u_\sigma(\mathbf{r})\frac{1}{\rho}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \varphi} - v_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho}\right]\right\}dS$$

$$= \frac{1}{2}\left[v_a(\mathbf{r}')\hat{\varphi}' + h_a(\mathbf{r}')\hat{z}\right], \quad \mathbf{r}' \in S_a\,. \tag{4.10}$$

Since

$$\hat{\rho} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}\,, \quad \hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} \tag{4.11}$$

we can perform the vector operations in (4.10) to get, for the φ'-component,

$$-\int_{S_a} v_a(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}dS - \int_{S_b} v_b(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}dS$$

$$+2\int_{\sigma}\left[u_\sigma(\mathbf{r})\frac{1}{\rho}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \varphi} - v_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho}\right]dS = \frac{1}{2}v_a(\mathbf{r}')\,, \quad \mathbf{r}' \in S_a \tag{4.12}$$

and for the *z*-component

$$
\begin{aligned}
&-\int_{S_a}\left\{v_a(\mathbf{r})\sin(\varphi-\varphi')\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z}+h_a(\mathbf{r})\hat{\rho}'\cdot\nabla g_D(\mathbf{r},\mathbf{r}')\right\}dS \\
&-\int_{S_b}\left\{v_b(\mathbf{r})\sin(\varphi-\varphi')\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z}+h_b(\mathbf{r})\hat{\rho}'\cdot\nabla g_D(\mathbf{r},\mathbf{r}')\right\}dS \\
&-2\int_{\sigma}\left[\sin(\varphi-\varphi')v_\sigma(\mathbf{r})-\cos(\varphi-\varphi')u_\sigma(\mathbf{r})\right]\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}dS=\frac{1}{2}h_a(\mathbf{r}')\,,\quad \mathbf{r}'\in S_a\,.
\end{aligned}
\tag{4.13}
$$

Similar expression is obtained from the second of (3.2) ($\mathbf{r}'\in S_b$, $\hat{n}'=-\hat{\rho}'$)

$$
\begin{aligned}
&\int_{S_a}v_a(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial\rho}dS+\int_{S_b}v_b(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial\rho}dS \\
&-2\int_{\sigma}\left[u_\sigma(\mathbf{r})\frac{1}{\rho}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial\varphi}-v_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial\rho}\right]dS=\frac{1}{2}v_b(\mathbf{r}')\,,\quad \mathbf{r}'\in S_b
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
&\int_{S_a}\left\{v_a(\mathbf{r})\sin(\varphi-\varphi')\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z}+h_a(\mathbf{r})\hat{\rho}'\cdot\nabla g_D(\mathbf{r},\mathbf{r}')\right\}dS \\
&+\int_{S_b}\left\{v_b(\mathbf{r})\sin(\varphi-\varphi')\frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial z}+h_b(\mathbf{r})\hat{\rho}'\cdot\nabla g_D(\mathbf{r},\mathbf{r}')\right\}dS \\
&+2\int_{\sigma}\left[\sin(\varphi-\varphi')v_\sigma(\mathbf{r})-\cos(\varphi-\varphi')u_\sigma(\mathbf{r})\right]\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}dS=\frac{1}{2}h_b(\mathbf{r}')\,,\quad \mathbf{r}'\in S_b\,.
\end{aligned}
\tag{4.15}
$$

We return to (4.6) and we write it in terms of its components

$$
\begin{aligned}
&-\int_{S_a}\left[v_a(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\sin(\varphi-\varphi')+h_a(\mathbf{r})\hat{\rho}'\cdot\nabla_t g(\mathbf{r},\mathbf{r}')\right]dS \\
&-\int_{S_b}\left[v_b(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\sin(\varphi-\varphi')+h_b(\mathbf{r})\hat{\rho}'\cdot\nabla_t g(\mathbf{r},\mathbf{r}')\right]dS=u_\sigma(\mathbf{r}')\,,\ \mathbf{r}'\in\sigma
\end{aligned}
\tag{4.16}
$$

and

$$\int_{S_a}\left[v_a(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\cos(\varphi-\varphi')-h_a(\mathbf{r})\hat{\varphi}'\cdot\nabla_t g(\mathbf{r},\mathbf{r}')\right]dS$$
$$+\int_{S_b}\left[v_b(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\cos(\varphi-\varphi')-h_b(\mathbf{r})\hat{\varphi}'\cdot\nabla_t g(\mathbf{r},\mathbf{r}')\right]dS=v_\sigma(\mathbf{r}')\,,\ \mathbf{r}'\in\sigma\,. \tag{4.17}$$

The system of integral equations is given by (4.12) through (4.17).

5. SIMPLIFICATION OF SCALAR INTEGRAL EQUATIONS

We recall the definition of the scalar Green's function

$$g(\mathbf{r},\mathbf{r}') = -\frac{\mathrm{e}^{-ikR}}{4\pi R}\,, \quad R = \left|\mathbf{r}-\mathbf{r}'\right| = \left[\rho^2 - 2\rho\rho'\cos(\varphi-\varphi') + \rho'^2 + (z-z')^2\right]^{\frac{1}{2}}. \tag{5.1}$$

We observe that

$$\int_{\varphi'}^{\varphi'+2\pi} \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \varphi'}\,d\varphi = -\int_{\varphi'}^{\varphi'+2\pi} \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \varphi}\,d\varphi = 0\,. \tag{5.2}$$

Since the various current density functions are independent of φ, the second term in the two integrals in (4.17) is zero; thus, we can write

$$\int_{S_a} v_a(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\cos(\varphi-\varphi')dS + \int_{S_b} v_b(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\cos(\varphi-\varphi')dS = v_\sigma(\mathbf{r}')\,,\ \mathbf{r}'\in\sigma\,. \tag{5.3}$$

This statement makes good physical sense because it says that the angular linear current density on σ depends only on the angular linear current densities on the walls of the coaxial line and not on the z-directed ones.

We encounter the same kind of term in (4.12), and we rewrite that expression as

$$-\int_{S_a} v_a(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}dS - \int_{S_b} v_b(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}dS - 2\int_{\sigma} v_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho}dS = \frac{1}{2}v_a(\mathbf{r}')\,, \quad \mathbf{r}'\in S_a\,. \tag{5.4}$$

A similar expression results from (4.14)

$$\int_{S_a} v_a(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}dS + \int_{S_b} v_b(\mathbf{r})\frac{\partial g_N(\mathbf{r},\mathbf{r}')}{\partial \rho}dS + 2\int_{\sigma} v_\sigma(\mathbf{r})\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho}dS = \frac{1}{2}v_b(\mathbf{r}')\,,\ \mathbf{r}'\in S_b\,. \tag{5.5}$$

From (5.1), we see that the scalar Green's function in cylindrical coordinates is an even function of $\varphi-\varphi'$ about zero. Thus, its Fourier series contains only cosine terms; in fact (reference 3, p. 487)

$$g(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\sum_{l=0}^{\infty}\varepsilon_l\cos l(\varphi-\varphi')\int_0^\infty J_l(\alpha\rho)J_l(\alpha\rho')\,\mathrm{e}^{-|z-z'|\sqrt{\alpha^2-k^2}}\ \frac{\alpha\,d\alpha}{\sqrt{\alpha^2-k^2}} \tag{5.6}$$

where ε_l is the Neumann symbol

$$\varepsilon_0 = 1\,; \quad \varepsilon_l = 2\,, \quad l = 1, 2, \ldots \tag{5.7}$$

Because of this, the terms in (4.16) containing the sine function dropout and we get

$$\int_{S_a} h_a(\mathbf{r}) \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS + \int_{S_b} h_b(\mathbf{r}) \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS = u_\sigma(\mathbf{r}')\,,\ \mathbf{r}' \in \sigma\,. \tag{5.8}$$

Again, this statement makes good physical sense because it says that the radial component of the linear current density on σ depends only on the z-component of the linear current density on the walls of the coaxial line. We can visualize the current coming up in the inner conductor, displacing itself radially at the termination of the line, and returning via the outer conductor. Similarly, from (4.13) we write

$$\begin{aligned}\int_{S_a} h_a(\mathbf{r}) \frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS + \int_{S_b} h_b(\mathbf{r}) \frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS \\ + 2\int_{\sigma} u_\sigma(\mathbf{r}) \cos(\varphi - \varphi') \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} dS = \frac{1}{2} h_a(\mathbf{r}')\,,\ \mathbf{r}' \in S_a\end{aligned} \tag{5.9}$$

and from (4.15)

$$\begin{aligned}-\int_{S_a} h_a(\mathbf{r}) \frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS - \int_{S_b} h_b(\mathbf{r}) \frac{\partial g_D(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS \\ - 2\int_{\sigma} u_\sigma(\mathbf{r}) \cos(\varphi - \varphi') \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} dS = \frac{1}{2} h_b(\mathbf{r}')\,,\ \mathbf{r}' \in S_b\,.\end{aligned} \tag{5.10}$$

From definitions (4.2), (4.3), and (4.4), we see that the last three equations involve both unknown and known quantities. By contrast, (5.3), (5.4), and (5.5) involve only unknown quantities, i.e., there is no generator that excites these circumferential currents. Indeed, a quick examination of the geometry of our problem and the way we excite the coaxial line reveal that there is no reason for these currents to exist; therefore, in (4.2). (4.3), and (4.4), we set all $C_n = 0$. From (2.14), and the discussion in Section 2, we conclude that the arrangement we have supports a TEM and a TM wave, but it does not support a TE wave.

6. SOLUTION OF THE SYSTEM OF INTEGRAL EQUATIONS

The system of integral equations consists of (5.8), (5.9), and (5.10). The three scalar functions that appear in them are defined in (4.2), (4.3), and (4.4) and involve an infinite number of unknowns. We can use the method of weighted residuals (reference 6) to convert each equation into an infinite system of linear algebraic equations. Thus, we will have three separate systems of equations, each involving the same unknowns. If we truncate the three systems into finite systems of $N+1$ equations in $N+1$ unknowns, then, as N tends to infinity, the solution for each system must converge to the same value; thus, we only need to numerically solve one of the three integral equations. We chose to deal with (5.8). We will do this in Part 3 of this report. In the next section, we derive far-field expressions.

7. FAR-FIELD REPRESENTATIONS

We derive a far-field expression for the geometry of this section. From Green's second identity (reference 7, p. 509), we have that in the upper-half space

$$\int_{D^+} \left\{ \mathbf{H}^r(\mathbf{r}) \cdot \left[\nabla \times \nabla \times \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right] - \left[\nabla \times \nabla \times \mathbf{H}^r(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right\} dV$$
$$= \int_{z=0} \left\{ \mathbf{H}^r(\mathbf{r}) \cdot \left[\hat{n} \times \nabla \times \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right] - \left[\hat{n} \times \nabla \times \mathbf{H}^r(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right\} dS \tag{7.1}$$

where $\mathbf{H}^r$ denotes the radiated magnetic field (total magnetic field in upper-half space) and $\boldsymbol{\Gamma}_2$ is the second-kind Green's dyadic. The latter satisfies the differential equation

$$\nabla \times \nabla \times \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') - k^2 \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') = ik \nabla \times \left[\delta(\mathbf{r},\mathbf{r}')\mathbf{I} - \delta(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i \right] \tag{7.2}$$

where $\mathbf{I}$ is the identity dyadic and $\mathbf{I}_i$ its image about the $z = 0$, and the boundary condition

$$\hat{z} \times \nabla \times \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') = \mathbf{0}\,, \quad z = 0\,. \tag{7.3}$$

Following standard procedures, we can show that

$$\int_{D^+} \left\{ \mathbf{H}^r(\mathbf{r}) \cdot \left[\nabla \times \nabla \times \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right] - \left[\nabla \times \nabla \times \mathbf{H}^r(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right\} dV$$
$$= ik \nabla' \times \mathbf{H}^r(\mathbf{r}') = -k^2 Y \mathbf{E}^r(\mathbf{r}')\,, \quad z' > 0\,. \tag{7.4}$$

Also,

$$\int_{z=0} \left\{ \mathbf{H}^r(\mathbf{r}) \cdot \left[\hat{n} \times \nabla \times \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right] - ikY \left[\hat{n} \times \mathbf{E}^r(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right\} dS$$
$$= -ikY \int_{\sigma} \left\{ \left[\hat{z} \times \mathbf{E}^r(\mathbf{r}) \right] \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') \right\} dS = -ikY \int_{\sigma} \mathbf{M}_{\sigma}(\mathbf{r}) \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') dS\,. \tag{7.5}$$

From the last two equations, we have that

$$\mathbf{E}^r(\mathbf{r}') = \frac{i}{k} \int_{\sigma} \mathbf{M}_{\sigma}(\mathbf{r}) \cdot \boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') dS\,, \quad z' > 0\,. \tag{7.6}$$

The dyadic in this expression is given by

$$\boldsymbol{\Gamma}_2(\mathbf{r},\mathbf{r}') = -ik \nabla \times \left[g(\mathbf{r},\mathbf{r}')\mathbf{I} - g(\mathbf{r},\mathbf{r}_i')\mathbf{I}_i \right] \tag{7.7}$$

Where

$$g(\mathbf{r},\mathbf{r}') = -\frac{\mathrm{e}^{-ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{7.8}$$

With this, we have that

$$\begin{aligned}
&\left\{\mathbf{M}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}') - \left[\mathbf{M}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}_i')\right]\cdot\mathbf{I}_i\right\}\Big|_{z=0} \\
&= \mathbf{M}_\sigma(\mathbf{r})\times\nabla_t g(\mathbf{r},\mathbf{r}') - \left[\mathbf{M}_\sigma(\mathbf{r})\times\nabla_t g(\mathbf{r},\mathbf{r}')\right]\cdot\mathbf{I}_i + \mathbf{M}_\sigma(\mathbf{r})\times\hat{z}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} + \left[\mathbf{M}_\sigma(\mathbf{r})\times\hat{z}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z}\right]\cdot\mathbf{I}_i \\
&= 2\mathbf{M}_\sigma(\mathbf{r})\times\nabla_t g(\mathbf{r},\mathbf{r}') + 2\mathbf{M}_\sigma(\mathbf{r})\times\hat{z}\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial z} = 2\mathbf{M}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')\Big|_{z=0}
\end{aligned} \tag{7.9}$$

which we substitute in (7.6) to get

$$\mathbf{E}^r(\mathbf{r}') = 2\int_\sigma \mathbf{M}_\sigma(\mathbf{r})\times\nabla g(\mathbf{r},\mathbf{r}')dS\,, \quad z'>0\,. \tag{7.10}$$

Similarly, from (2.13) of Chapter 1 of Part 1, we have

$$\mathbf{H}^r\left(\mathbf{r}'\right) = \frac{2}{ikZ}\int_\sigma \mathbf{M}_\sigma(\mathbf{r})\cdot\nabla\nabla g(\mathbf{r},\mathbf{r}')dS - i2kY\int_\sigma \mathbf{M}_\sigma(\mathbf{r})g(\mathbf{r},\mathbf{r}')dS\,, \quad z'>0\,. \tag{7.11}$$

This equation can also be obtained by applying one of Maxwell’s equations (Ampère’s law) to (7.10).

In the far field ($r'\to\infty$)

$$g(\mathbf{r},\mathbf{r}') \sim -\frac{\mathrm{e}^{-ik(r'-\mathbf{r}\cdot\hat{r}')}}{4\pi r'}\,, \quad \nabla g(\mathbf{r},\mathbf{r}') \sim ik\frac{\mathrm{e}^{-ik(r'-\mathbf{r}\cdot\hat{r}')}}{4\pi r'}\hat{r}'\,, \quad \nabla\nabla g(\mathbf{r},\mathbf{r}') \sim k^2\frac{\mathrm{e}^{-ik(r'-\mathbf{r}\cdot\hat{r}')}}{4\pi r'}\hat{r}'\hat{r}'\,. \tag{7.12}$$

Using this in the previous two equations, we get

$$\mathbf{E}^r(\mathbf{r}') = -\frac{ik\,\mathrm{e}^{-ikr'}}{2\pi r'}\hat{r}'\times\int_\sigma \mathbf{M}_\sigma(\mathbf{r})\,\mathrm{e}^{ik\mathbf{r}\cdot\hat{r}'}\,dS\,, \quad r'\to\infty \tag{7.13}$$

and

$$\mathbf{H}^r\left(\mathbf{r}'\right) = -\frac{ikY\,\mathrm{e}^{-ikr'}}{2\pi r'}\hat{r}'\times\left[\hat{r}'\times\int_\sigma \mathbf{M}_\sigma(\mathbf{r})\,\mathrm{e}^{ik\mathbf{r}\cdot\hat{r}'}\,dS\right], \quad r'\to\infty\,. \tag{7.14}$$

These are the far-field representations.

We will attempt to compute the integral in these expressions. From (1.2) and (2.14)

$$\mathbf{M}_\sigma(\mathbf{r}) = \mathbf{M}_\sigma(\rho,\varphi,0) = \hat{z}\times\mathbf{E}^r(\rho,\varphi,0) = \hat{z}\times\mathbf{E}^t(\rho,\varphi,0)$$
$$= V\left[\frac{1+A}{\ln(\chi)\rho} - i\sum_{n=1}^{\infty} B_n\lambda_{0n}P_{0n}{}'(\rho)\right]\hat{\varphi}\,. \tag{7.15}$$

We substitute in the integral

$$\mathbf{I} = \int_\sigma \mathbf{M}_\sigma(\mathbf{r})\,\mathrm{e}^{ik\mathbf{r}\cdot\hat{r}'}\,dS = V\int_a^b \rho d\rho\left[\frac{1+A}{\ln(\chi)\rho} - i\sum_{n=1}^{\infty} B_n\lambda_{0n}P_{0n}{}'(\rho)\right]\int_0^{2\pi} d\varphi\hat{\varphi}\,\mathrm{e}^{ik\mathbf{r}\cdot\hat{r}'}\,. \tag{7.16}$$

For the second integral, we write

$$\mathbf{I}_1 = \int_0^{2\pi} d\varphi\hat{\varphi}\,\mathrm{e}^{ik\mathbf{r}\cdot\hat{r}'} = \int_0^{2\pi}(-\hat{x}\sin\varphi + \hat{y}\cos\varphi)\,\mathrm{e}^{ik\sin\vartheta'\rho\cos(\varphi-\varphi')}\,d\varphi$$
$$= \int_{-\varphi'}^{2\pi-\varphi'}\left[-\hat{x}\sin(\psi+\varphi') + \hat{y}\cos(\psi+\varphi')\right]\mathrm{e}^{ik\sin\vartheta'\rho\cos\psi}\,d\psi$$
$$= -\hat{\rho}'\int_{-\varphi'}^{2\pi-\varphi'}\sin\psi\,\mathrm{e}^{ik\sin\vartheta'\rho\cos\psi}\,d\psi + \hat{\varphi}'\int_{-\varphi'}^{2\pi-\varphi'}\cos\psi\,\mathrm{e}^{ik\sin\vartheta'\rho\cos\psi}\,d\psi = \hat{\varphi}' i2\pi J_1(k\sin\vartheta'\rho) \tag{7.17}$$

and substitute in (7.16)

$$\mathbf{I} = \hat{\varphi}' i2\pi V\int_a^b J_1(k\sin\vartheta'\rho)\left[\frac{1+A}{\ln(\chi)\rho} - i\sum_{n=1}^{\infty} B_n\lambda_{0n}P_{0n}{}'(\rho)\right]\rho d\rho$$
$$= \hat{\varphi}' i2\pi V\left\{\frac{1+A}{\ln(\chi)}\int_a^b J_1(k\sin\vartheta'\rho)d\rho - i\sum_{n=1}^{\infty} B_n\lambda_{0n}\int_a^b J_1(k\sin\vartheta'\rho)P_{0n}{}'(\rho)\rho d\rho\right\}. \tag{7.18}$$

For the first of the last two integrals, we write

$$I_1 = \int_a^b J_1(k\sin\vartheta'\rho)d\rho = -\frac{1}{k\sin\vartheta'}\int_{ka\sin\vartheta'}^{kb\sin\vartheta'} J_0{}'(u)du = \frac{J_0(ka\sin\vartheta') - J_0(kb\sin\vartheta')}{k\sin\vartheta'} \tag{7.19}$$

while for the second

$$I_2 = \int_a^b J_1(k\sin\vartheta'\rho)P_{0n}{}'(\rho)\rho d\rho = -\nu_{0n}\int_a^b J_1(k\sin\vartheta'\rho)\left[J_1(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]\rho d\rho$$
$$= -\frac{\nu_{0n}\rho}{\nu_{0n}{}^2 - k^2\sin^2\vartheta'}\left[\nu_{0n}J_2(\nu_{0n}\rho)J_1(k\sin\vartheta'\rho) - k\sin\vartheta' J_1(\nu_{0n}\rho)J_2(k\sin\vartheta'\rho)\right]\Big|_a^b$$
$$+\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\frac{\nu_{0n}\rho}{\nu_{0n}{}^2 - k^2\sin^2\vartheta'}\left[\nu_{0n}Y_2(\nu_{0n}\rho)J_1(k\sin\vartheta'\rho) - kY_1(\nu_{0n}\rho)J_2(k\sin\vartheta'\rho)\right]\Big|_a^b$$

$$
\begin{aligned}
&= -\frac{\nu_{0n}{}^2 \rho}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_2(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_2(\nu_{0n}\rho) \right] J_1(k \sin \vartheta' \rho) \Big|_a^b \\
&+ \frac{\nu_{0n} k \sin \vartheta' \rho}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}\rho) \right] J_2(k \sin \vartheta' \rho) \Big|_a^b \\
&= -\frac{2\nu_{0n}}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}\rho) \right] J_1(k \sin \vartheta' \rho) \Big|_a^b \\
&+ \frac{\nu_{0n} k \sin \vartheta' \rho}{\nu_{0n}{}^2 - k^2 \sin \vartheta'^2} \left[J_1(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}\rho) \right] J_2(k \sin \vartheta' \rho) \Big|_a^b \\
&= -\frac{\nu_{0n} k \sin \vartheta' \rho}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}\rho) \right] \left[\frac{2J_1(k \sin \vartheta' \rho)}{k \sin \vartheta' \rho} - J_2(k \sin \vartheta' \rho) \right] \Big|_a^b \\
&= -\frac{\nu_{0n} k \sin \vartheta' \rho}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}\rho) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}\rho) \right] J_0(k \sin \vartheta' \rho) \Big|_a^b \\
&= -\frac{\nu_{0n} k b \sin \vartheta'}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}b) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}b) \right] J_0(kb \sin \vartheta') \\
&+ \frac{\nu_{0n} k a \sin \vartheta'}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}a) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}a) \right] J_0(ka \sin \vartheta') \\
&= -\frac{\nu_{0n} k b \sin \vartheta'}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}b) - \frac{J_0(\nu_{0n}b)}{Y_0(\nu_{0n}b)} Y_1(\nu_{0n}b) \right] J_0(kb \sin \vartheta') \\
&+ \frac{\nu_{0n} k a \sin \vartheta'}{\nu_{0n}{}^2 - k^2 \sin^2 \vartheta'} \left[J_1(\nu_{0n}a) - \frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)} Y_1(\nu_{0n}a) \right] J_0(ka \sin \vartheta') \\
&= -\frac{2k \sin \vartheta'}{\pi(\nu_{0n}{}^2 - k^2 \sin^2 \vartheta')} \frac{J_0(kb \sin \vartheta')}{Y_0(\nu_{0n}b)} + \frac{2k \sin \vartheta'}{\pi(\nu_{0n}{}^2 - k^2 \sin^2 \vartheta')} \frac{J_0(ka \sin \vartheta')}{Y_0(\nu_{0n}a)} \\
&= \frac{2k \sin \vartheta'}{\pi(\nu_{0n}{}^2 - k^2 \sin^2 \vartheta')} \left[\frac{J_0(ka \sin \vartheta')}{Y_0(\nu_{0n}a)} - \frac{J_0(kb \sin \vartheta')}{Y_0(\nu_{0n}b)} \right].
\end{aligned}
\tag{7.20}
$$

We substitute the last two equations in (7.18) and use (2.6)

$$
\begin{aligned}
\mathbf{I} &= \hat{\varphi}' i 2\pi V \left\{ \frac{1+A}{\ln(\chi)} \frac{J_0(ka \sin \vartheta') - J_0(kb \sin \vartheta')}{k \sin \vartheta'} \right. \\
&\left. - i \sum_{n=1}^{\infty} B_n \lambda_{0n} \frac{2k \sin \vartheta'}{\pi(\nu_{0n}{}^2 - k^2 \sin^2 \vartheta')} \left[\frac{J_0(ka \sin \vartheta')}{Y_0(\nu_{0n}a)} - \frac{J_0(kb \sin \vartheta')}{Y_0(\nu_{0n}b)} \right] \right\} \\
&= \hat{\varphi}' i 2\pi V \left\{ \frac{1+A}{\ln(\chi)} \frac{J_0(ka \sin \vartheta') - J_0(kb \sin \vartheta')}{k \sin \vartheta'} \right.
\end{aligned}
$$

$$+\frac{2k\sin\vartheta'}{\pi}\sum_{n=1}^{\infty}B_n\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\nu_{0n}{}^2-k^2\sin^2\vartheta'}\left[\frac{J_0(ka\sin\vartheta')}{Y_0(\nu_{0n}a)}-\frac{J_0(kb\sin\vartheta')}{Y_0(\nu_{0n}b)}\right]\Bigg\}. \tag{7.21}$$

From this and (7.13)

$$\mathbf{E}^r(\mathbf{r})=-\hat{\vartheta}\frac{\mathrm{e}^{-ikr}}{r}Vk\left\{\frac{1+A}{\ln(\chi)}\frac{J_0(ka\sin\vartheta)-J_0(kb\sin\vartheta)}{k\sin\vartheta}\right.$$

$$\left.+\frac{2k\sin\vartheta}{\pi}\sum_{n=1}^{\infty}B_n\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\nu_{0n}{}^2-k^2\sin^2\vartheta}\left[\frac{J_0(ka\sin\vartheta)}{Y_0(\nu_{0n}a)}-\frac{J_0(kb\sin\vartheta)}{Y_0(\nu_{0n}b)}\right]\right\},\quad r\to\infty \tag{7.22}$$

while from (7.14)

$$\mathbf{H}^r\left(\mathbf{r}\right)=-\hat{\varphi}\frac{\mathrm{e}^{-ikr}}{r}YVk\left\{\frac{1+A}{\ln(\chi)}\frac{J_0(ka\sin\vartheta)-J_0(kb\sin\vartheta)}{k\sin\vartheta}\right.$$

$$\left.+\frac{2k\sin\vartheta}{\pi}\sum_{n=1}^{\infty}B_n\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\nu_{0n}{}^2-k^2\sin^2\vartheta}\left[\frac{J_0(ka\sin\vartheta)}{Y_0(\nu_{0n}a)}-\frac{J_0(kb\sin\vartheta)}{Y_0(\nu_{0n}b)}\right]\right\},\quad r\to\infty \tag{7.23}$$

where, in these two expressions, we changed from primed to unprimed coordinates. These are the far-field representations of the electric and the magnetic field in terms of the unknown coefficients of the current density expansions.

8. CONCLUSIONS

For a coaxial line, as shown in Figure 1.1, we derived three BIEs. We also wrote the unknown current densities as expansions in the modes of the coaxial line. We concluded in Section 6 that we need deal with only one of the integral equations to numerically determine the coefficients of these expansions. This will be explicitly done in the parts that follow. In Section 7, we present the expansions of the far fields.

APPENDIX: THE ELECTROMAGNETIC FIELDS OF A COAXIAL TRANSMISSION LINE

A1. INTRODUCTION

In this note, we develop the eigenvalues and eigenfunctions for an infinite coaxial line. The inner and outer walls of the line are perfectly conducting, with radii a and b, respectively (see Figure A1). The axis of the line is the z-axis. Derivations can be found in many books. We have consulted Jones (reference 8), Stratton (reference 9), and Tai (reference 10). The time dependence of the electromagnetic fields is $+i\omega t$. Thus, Maxwell's equations are

$$\nabla \times \mathbf{E} = -ikZ\mathbf{H}\,, \quad \nabla \times \mathbf{H} = ikY\mathbf{E}\,, \quad k = \omega\sqrt{\mu\varepsilon}\,, \quad Z = Y^{-1} = \sqrt{\mu/\varepsilon} \tag{A1.1}$$

where μ and ε are the constitutive parameters of the space between the two conductors. On C_a and C_b, the tangential component of the electric field is zero

$$\hat{n} \times \mathbf{E} = \mathbf{0}\,. \tag{A1.2}$$

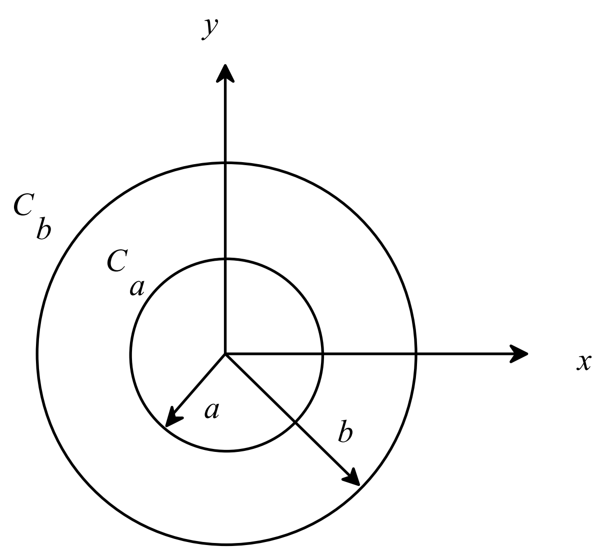

Figure A1. Cross section of coaxial transmission line. C_a and C_b represent the inner and outer circles. The region between the two conductors is denoted by D.

A2. GENERATION OF WAVES IN THE COAXIAL LINE

It is well known (reference 9) that, in a source-free region, electromagnetic fields can be expressed in terms of two scalar functions. These functions are components of Hertz vectors of arbitrary direction. In this case, we take the direction to be that of the z-axis and write

$$\mathbf{\Pi}(\mathbf{r}) = \Pi(\mathbf{r})\hat{z}\,, \quad \mathbf{M}(\mathbf{r}) = M(\mathbf{r})\hat{z} \tag{A2.1}$$

where

$$\nabla^2 \Pi(\mathbf{r}) + k^2 \Pi(\mathbf{r}) = 0, \quad \nabla^2 M(\mathbf{r}) + k^2 M(\mathbf{r}) = 0\,. \tag{A2.2}$$

The fields generated by these Hertzian potentials are (reference 9)

$$\mathbf{E}(\mathbf{r}) = \nabla\times\nabla\times[\Pi(\mathbf{r})\hat{z}] - ikZ\nabla\times[M(\mathbf{r})\hat{z}]\,, \quad \mathbf{H}(\mathbf{r}) = \nabla\times\nabla\times[M(\mathbf{r})\hat{z}] + ikY\nabla\times[\Pi(\mathbf{r})\hat{z}]\,. \tag{A2.3}$$

A3. TRANSVERSE MAGNETIC WAVES

If in (A2.1) we let $M(\mathbf{r}) \equiv 0$, we find that the magnetic field does not have a component along the z-axis. The z-component of the electric field is given by

$${}^{TM}E^z(\mathbf{r}) = \hat{z}\cdot\nabla\times\nabla\times[\Pi(\mathbf{r})\hat{z}] = \frac{\partial^2 \Pi(\mathbf{r})}{\partial z^2} + k^2 \Pi(\mathbf{r})\,. \tag{A3.1}$$

From (A1.2), we have that ${}^{TM}E^z$ must be equal to zero when $\rho = a$ or $\rho = b$. For this, it is sufficient that

$$\Pi(a,\varphi,z) = \Pi(b,\varphi,z) = 0\,. \tag{A3.2}$$

For, if this is the case, it follows from the Hugoniot-Hadamard theorem (reference 11) that the second derivative with respect to z is also zero there. Thus, from (A2.2) and (A3.2), we have the boundary-value problem

$$\nabla^2 \Pi(\mathbf{r}) + k^2 \Pi(\mathbf{r}) = 0, \quad \Pi(a,\varphi,z) = \Pi(b,\varphi,z) = 0\,. \tag{A3.3}$$

We use cylindrical coordinates (ρ,φ,z) and write

$$\Pi(\rho,\varphi,z) = f(\rho,\varphi)h(z)\,. \tag{A3.4}$$

Substitution in (A3.3) gives

$$-\frac{\nabla_t{}^2 f(\rho,\varphi)}{f(\rho,\varphi)} = \frac{h''(z)}{h(z)} + k^2 = \nu^2 \tag{A3.5}$$

where ∇_t is the transverse component of the Laplacian and ν^2 is a constant that will be determined from the boundary condition. Expressing the Laplacian in cylindrical coordinates, we have

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial\varphi^2} + \nu^2\rho^2 f(\rho,\varphi) = 0\,. \tag{A3.6}$$

Writing,

$$f(\rho,\varphi) = P(\rho)\Phi(\varphi) \tag{A3.7}$$

we obtain

$$\frac{\rho \dfrac{d}{d\rho}\left(\rho \dfrac{dP(\rho)}{d\rho}\right)}{P(\rho)} + \nu^2\rho^2 = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = m^2 \tag{A3.8}$$

where the constant m is an integer since the solution in the angular direction has period 2π. The function Φ is then of the form

$$\Phi_{m\,{}^{e}_{o}}(\varphi) = \begin{cases} \cos m\varphi \\ \sin m\varphi \end{cases}, \quad m = 0,1,2,\dots \; . \tag{A3.9}$$

The remaining equation in (A3.8) is Bessel's equation

$$\rho \frac{d}{d\rho}\left(\rho \frac{dP(\rho)}{d\rho}\right) + \left(\nu^2\rho^2 - m^2\right)P(\rho) = 0 \tag{A3.10}$$

and its solution is the linear combination of a Bessel and a Neumann function

$$P_m(\rho) = a_m J_m(\nu_m\rho) + b_m Y_m(\nu_m\rho)\, . \tag{A3.11}$$

These are the functions that also must satisfy the boundary conditions

$$P_m(a) = P_m(b) = 0\, . \tag{A3.12}$$

This results in the homogeneous system of equations

$$\begin{aligned} a_m J_m(\nu_m a) + b_m Y_m(\nu_m a) &= 0 \\ a_m J_m(\nu_m b) + b_m Y_m(\nu_m b) &= 0 \end{aligned} \tag{A3.13}$$

which has non-trivial solutions only if its determinant is zero

$$J_m(\nu_m a)Y_m(\nu_m b) - J_m(\nu_m b)Y_m(\nu_m a) = 0\, . \tag{A3.14}$$

This is a transcendental equation with roots ν_{mn}, $n = 1,2,\dots$. For these roots, the solutions of the system are

$$b_{mn} = -\frac{J_m(\nu_{mn}a)}{Y_m(\nu_{mn}a)}a_{mn} = -\frac{J_m(\nu_{mn}b)}{Y_m(\nu_{mn}b)}a_{mn}\,. \tag{A3.15}$$

For (A3.11) we can then write

$$P_{mn}(\rho) = J_m(\nu_{mn}\rho) - \frac{J_m(\nu_{mn}a)}{Y_m(\nu_{mn}a)}Y_m(\nu_{mn}\rho) \tag{A3.16}$$

and for (A3.7)

$$f_{mn{}^{e}_{o}}(\rho,\varphi) = P_{mn}(\rho)\Phi_{m{}^{e}_{o}}(\varphi)\,. \tag{A3.17}$$

These functions are orthogonal, in the sense that

$$\int_D f_{mn{}^{e}_{o}}(\rho,\varphi) f_{m'n{}^{e}_{o}}(\rho,\varphi)dS = A_{mm}\delta_{mm'}\ , \quad \int_D f_{mn{}^{e}_{o}}(\rho,\varphi) f_{m'n{}^{o}_{e}}(\rho,\varphi)dS = 0\,. \tag{A3.18}$$

They can also be orthonormalized.

Returning to (A3.5), we write

$$h''(z) + (k^2 - \nu_{mn}{}^2)h(z) = 0 \tag{A3.19}$$

which has solutions

$$h^{\pm}_{mn}(z) = \mathrm{e}^{\pm i\lambda_{mn}z} \tag{A3.20}$$

where

$$\lambda_{mn} = \begin{cases} \sqrt{k^2 - \nu_{mn}{}^2}\ , & k^2 > \nu_{mn}{}^2 \\ -i\sqrt{\nu_{mn}{}^2 - k^2}\ , & \nu_{mn}{}^2 > k^2 \end{cases}. \tag{A3.21}$$

For the upper root, the solution with the plus sign represents a wave traveling along the negative z-axis while the one with the minus sign represents one that travels along the positive z-axis. For the lower, we have exponential decay along these directions. From (A3.4), (A3.17), and (A3.20), we can write,

$$\Pi^{\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z) = \mathrm{e}^{\pm i\lambda_{mn}z}\ P_{mn}(\rho)\Phi_{m{}^{e}_{o}}(\varphi)\,. \tag{A3.22}$$

These are the eigenfunctions that comprise the z-component of the electric Hertz potential. From this we can construct the electromagnetic field according to (A2.3). Thus, the eigenfunctions for the z-component of the electric field are

$$ {}^{TM}E^{z\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z) = \nu_{mn}{}^{2}\, \mathrm{e}^{\pm i\lambda_{mn}z}\, P_{mn}(\rho)\Phi_{m{}^{e}_{o}}(\varphi) \tag{A3.23} $$

while those for the ρ- and φ–components

$$ {}^{TM}E^{\rho\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z) = \frac{\partial^2 \Pi^{\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z)}{\partial\rho\partial z} = \pm i\lambda_{mn}\, \mathrm{e}^{\pm i\lambda_{mn}z}\, P_{mn}{}'(\rho)\Phi_{m{}^{e}_{o}}(\varphi) \tag{A3.24} $$

and

$$ {}^{TM}E^{\varphi\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z) = \frac{\partial^2 \Pi^{\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z)}{\rho\partial\varphi\partial z} = \pm \frac{i\lambda_{mn}}{\rho}\, \mathrm{e}^{\pm i\lambda_{mn}z}\, P_{mn}(\rho)\Phi_{m{}^{e}_{o}}{}'(\varphi)\,. \tag{A3.25} $$

We note that the φ–component automatically satisfies the boundary conditions. For the magnetic field

$$ {}^{TM}H^{\rho\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z) = \frac{ikY}{\rho}\frac{\partial \Pi^{\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z)}{\partial\varphi} = \frac{ikY}{\rho}\, \mathrm{e}^{\pm i\lambda_{mn}z}\, P_{mn}(\rho)\Phi_{m{}^{e}_{o}}{}'(\varphi) \tag{A3.26} $$

and

$$ {}^{TM}H^{\varphi\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z) = ikY\frac{\partial \Pi^{\pm}_{mn{}^{e}_{o}}(\rho,\varphi,z)}{\partial\rho} = ikY\, \mathrm{e}^{\pm i\lambda_{mn}z}\, P_{mn}{}'(\rho)\Phi_{m{}^{e}_{o}}(\varphi)\,. \tag{A3.27} $$

Again, we see that (A3.26) satisfies the boundary condition that the component of the magnetic field normal to the perfectly conducting surfaces must be zero there. This condition follows from (A1.2).

A4. TRANSVERSE ELECTRIC WAVES

If in (A2.1) we let $\Pi(\mathbf{r}) \equiv 0$, we find that the electric field does not have a component along the z-axis. The φ-component of the electric field is given by

$$ {}^{TE}E^{\varphi}(\mathbf{r}) = ikZ\,\frac{\partial M(\mathbf{r})}{\partial \rho}\,. \tag{A4.1} $$

For (A1.2) to be satisfied, it is sufficient that the normal derivative of the magnetic Hertz potential on the two conductors is zero; thus, we have the boundary-value problem

$$ \nabla^2 M(\mathbf{r}) + k^2 M(\mathbf{r}) = 0\,, \quad \frac{\partial M(a,\varphi,z)}{\partial a} = \frac{\partial M(b,\varphi,z)}{\partial b} = 0\,. \tag{A4.2} $$

Proceeding as in the TM case, we write

$$ M(\rho,\varphi,z) = g(\rho,\varphi)h(z) \tag{A4.3} $$

and obtain

$$ -\frac{\nabla_t^{\,2} g(\rho,\varphi)}{g(\rho,\varphi)} = \frac{h''(z)}{h(z)} + k^2 = \mu^2 \tag{A4.4} $$

where g satisfies

$$ \rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) + \frac{\partial^2 g}{\partial\varphi^2} + \mu^2\rho^2 g(\rho,\varphi) = 0\,. \tag{A4.5} $$

Letting

$$ g(\rho,\varphi) = R(\rho)\Phi(\varphi) \tag{A4.6} $$

where Φ is defined in (A3.9), we get that R satisfies Bessel's equation (A3.10) with ν replaced by μ; thus,

$$ R_m(\rho) = c_m J_m(\mu_m\rho) + d_m Y_m(\mu_m\rho)\,. \tag{A4.7} $$

From the boundary condition (A4.2),

$$ \begin{aligned} c_m J_m{}'(\mu_m a) + d_m Y_m{}'(\mu_m a) &= 0 \\ c_m J_m{}'(\mu_m b) + d_m Y_m{}'(\mu_m b) &= 0 \end{aligned} \tag{A4.8} $$

and from this we get the condition

$$ J_m{}'(\mu_m a)Y_m{}'(\mu_m b) - J_m{}'(\mu_m b)Y_m{}'(\mu_m a) = 0\,. \tag{A4.9} $$

This is a transcendental equation with roots μ_{mn}, $n = 1, 2, \ldots$. For these roots, the solutions of the system are

$$d_{mn} = -\frac{J_m{}'(\mu_{mn}a)}{Y_m{}'(\mu_{mn}a)} c_{mn} = -\frac{J_m{}'(\mu_{mn}b)}{Y_m{}'(\mu_{mn}b)} c_{mn} \tag{A4.10}$$

and for (A4.7) we write

$$R_{mn}(\rho) = J_m(\mu_{mn}\rho) - \frac{J_m{}'(\mu_{mn}a)}{Y_m{}'(\mu_{mn}a)} Y_m(\mu_{mn}\rho) . \tag{A4.11}$$

For (A4.6) then we write

$$g_{mn_o^e}(\rho,\varphi) = R_{mn}(\rho)\Phi_{m_o^e}(\varphi) . \tag{A4.12}$$

These functions are orthogonal in the sense of (A3.18) and can be orthonormalized.

As with (A3.19), the solutions of h in (A4.4) are given by

$$h_{mn}^{\pm}(z) = \mathrm{e}^{\pm i\kappa_{mn}z} \tag{A4.13}$$

where

$$\kappa_{mn} = \begin{cases} \sqrt{k^2 - \mu_{mn}{}^2}\,, & k^2 > \mu_{mn}{}^2 \\ -i\sqrt{\mu_{mn}{}^2 - k^2}\,, & \mu_{mn}{}^2 > k^2 \end{cases} . \tag{A4.14}$$

For the eigenfunctions of the magnetic potential in (A4.3), we can write

$$M^{\pm}_{mn_o^e}(\rho,\varphi,z) = \mathrm{e}^{\pm i\kappa_{mn}z} R_{mn}(\rho)\Phi_{m_o^e}(\varphi) . \tag{A4.15}$$

From (A3.23), the corresponding functions for the electromagnetic fields are

$$^{TE}H^{z\pm}_{mn_o^e}(\rho,\varphi,z) = \hat{z}\cdot\nabla\times\nabla\times[M^{\pm}_{mn_o^e}\hat{z}] = \frac{\partial^2 M^{\pm}_{mn_o^e}}{\partial z^2} + k^2 M^{\pm}_{mn_o^e} = \mu_{mn}{}^2\,\mathrm{e}^{\pm i\kappa_{mn}z} R_{mn}(\rho)\Phi_{m_o^e}(\varphi) \tag{A4.16}$$

$$ {}^{TE}H^{\rho\pm}_{mn\substack{e\\o}}(\rho,\varphi,z) = \frac{\partial^2 M^{\pm}_{mn\substack{e\\o}}}{\partial\rho\partial z} = \pm i\kappa_{mn}\,\mathrm{e}^{\pm i\kappa_{mn}z}\,R_{mn}'(\rho)\mathit{\Phi}_{m\substack{e\\o}}(\varphi) \tag{A4.17}$$

$$ {}^{TE}H^{\varphi\pm}_{mn\substack{e\\o}}(\rho,\varphi,z) = \frac{\partial^2 M^{\pm}_{mn\substack{e\\o}}}{\rho\partial\varphi\partial z} = \pm\frac{i\kappa_{mn}}{\rho}\,\mathrm{e}^{\pm i\kappa_{mn}z}\,R_{mn}(\rho)\mathit{\Phi}_{m\substack{e\\o}}'(\varphi) \tag{A4.18}$$

$$ {}^{TE}E^{\rho\pm}_{mn\substack{e\\o}}(\rho,\varphi,z) = -\frac{ikZ}{\rho}\frac{\partial M^{\pm}_{mn\substack{e\\o}}}{\partial\varphi} = -\frac{ikZ}{\rho}\,\mathrm{e}^{\pm i\kappa_{mn}z}\,R_{mn}(\rho)\mathit{\Phi}_{m\substack{e\\o}}'(\varphi) \tag{A4.19}$$

$$ {}^{TE}E^{\varphi\pm}_{mn\substack{e\\o}}(\rho,\varphi,z) = ikZ\,\frac{\partial M^{\pm}_{mn\substack{e\\o}}}{\partial\rho} = ikZ\,\mathrm{e}^{\pm i\kappa_{mn}z}\,R_{mn}'(\rho)\mathit{\Phi}_{m\substack{e\\o}}(\varphi)\,. \tag{A4.20}$$

We note that all boundary conditions are satisfied.

A5. TRANSVERSE ELECTROMAGNETIC WAVES

In this case, the z-component of the electric and magnetic fields is zero. From (A3.1) and (A4.16), this implies that the fields have a $\mathrm{e}^{\pm ikz}$ dependence. From (A2.3), the part of the electric field due to the electric Hertz potential is

$$\begin{aligned} {}^{TEM}\mathbf{E}(\mathbf{r}) &= \nabla\times\nabla\times[\mathit{\Pi}(\mathbf{r})\hat{z}] = \nabla\times[\nabla\mathit{\Pi}(\mathbf{r})\times\hat{z}] = -\hat{z}\nabla^2\mathit{\Pi}(\mathbf{r}) + \frac{\partial}{\partial z}[\nabla\mathit{\Pi}(\mathbf{r})] \\ &= \hat{z}\left[k^2\mathit{\Pi}(\mathbf{r}) + \frac{\partial^2\mathit{\Pi}(\mathbf{r})}{\partial z^2}\right] + \nabla_t\left[\frac{\partial\mathit{\Pi}(\mathbf{r})}{\partial z}\right]. \end{aligned} \tag{A5.1}$$

Since the z-component is zero, we have that

$$ {}^{TEM}\mathbf{E}(\mathbf{r}) = \nabla_t\left[\frac{\partial\mathit{\Pi}(\mathbf{r})}{\partial z}\right]. \tag{A5.2}$$

Write

$$ {}^{TEM}\mathbf{E}^{\pm}(\mathbf{r}) = \mathrm{e}^{\pm ikz}\,\mathbf{e}^{\pm}(\boldsymbol{\rho})\,,\quad \boldsymbol{\rho} = (x,y)\,. \tag{A5.3}$$

From this

$$ 0 = \nabla\cdot\left[{}^{TEM}\mathbf{E}^{\pm}(\mathbf{r})\right] = \nabla\left(\mathrm{e}^{\pm ikz}\right)\cdot\mathbf{e}^{\pm}(\boldsymbol{\rho}) + \mathrm{e}^{\pm ikz}\,\nabla\cdot\mathbf{e}^{\pm}(\boldsymbol{\rho}) = \mathrm{e}^{\pm ikz}\,\nabla\cdot\mathbf{e}^{\pm}(\boldsymbol{\rho}) \tag{A5.4}$$

so that

$$\nabla \cdot \mathbf{e}^{\pm}(\mathbf{r}) = 0 . \tag{A5.5}$$

Since in (A5.2) the electric field is expressible in terms of a scalar function, we can let that function be $\mathrm{e}^{\pm ikz} V^{\pm}(\boldsymbol{\rho})$, so that we can write for (A5.2)

$$\mathbf{e}^{\pm}(\boldsymbol{\rho}) = -\nabla V^{\pm}(\boldsymbol{\rho}) . \tag{A5.6}$$

From (A5.5)

$$\nabla^2 V^{\pm}(\boldsymbol{\rho}) = 0 . \tag{A5.7}$$

The general solution of this is

$$V^{\pm}_{\substack{e\\m\\o}}(\boldsymbol{\rho}) = U_m(\rho)\Phi_{\substack{e\\m\\o}}(\varphi) \tag{A5.8}$$

where the angular function is defined in (A3.9) while the radial one satisfies the differential equation

$$\rho^2 U_m''(\rho) + \rho U_m'(\rho) - m^2 U_m(\rho) = 0 . \tag{A5.9}$$

This is an Euler-type equation with solutions

$$U_0(\rho) = A_0 \ln \rho + B_0 \ ; \quad U_m(\rho) = A_m \rho^m + B_m \rho^{-m} \ , \quad m = 1, 2, \ldots \ . \tag{A5.10}$$

If we reconstruct the angular component of the electric field using these solutions, we find that, for the boundary conditions to be satisfied, we must have

$$\begin{array}{l} A_m a^m + B_m a^{-m} = 0 \\ A_m b^m + B_m b^{-m} = 0 \end{array} , \quad m = 1, 2, \ldots \ . \tag{A5.11}$$

This system has only the trivial solution unless $a = b$, a case of no interest; thus,

$$A_m = B_m = 0 \ , \quad m = 1, 2, \ldots \tag{A5.12}$$

and, for the potential, we get the simple expression

$$V^{\pm}(\boldsymbol{\rho}) = A_0 \ln \rho + B_0 . \tag{A5.13}$$

The unknown constants are determined by the potential difference between the two conductors. For example, if the inner conductor is at potential V and the outer at zero, then the boundary conditions give

$$\begin{aligned} A_0 \ln a + B_0 &= V \\ A_0 \ln b + B_0 &= 0 \end{aligned} \tag{A5.14}$$

from which we determine non-trivial values for the constants.

If we follow the same line of reasoning using the magnetic Hertz vector, we will find that we must have the radial derivative of the potential equal to zero on the two conductors. This will result in (A5.12) being true, but also that $A_0 = 0$; thus, the potential will be equal to a constant (B_0) and the fields will be equal to zero. We see then that, in the TEM case, the fields are determined from a single scalar function, the electric potential.

A7. CONCLUSION

We have derived the eigenfunctions that may be present in the space between the two perfect conductors of a coaxial line. The TEM mode will always be present, irrespective of frequency. The presence of other modes depends first and foremost on the feeding arrangement. If the fields of the source are independent of the angular direction, so will the total fields. If not, then a number of modes is possible, depending on the dimensions of the inner and outer conductor. We note that both in the TM and the TE case we have a term (m = 0) independent of φ.

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PART 3
NUMERICAL SOLUTION OF INTEGRAL EQUATION FOR COAXIAL LINE

ABSTRACT

This is the third part of the report on the formulation of the problem of radiation of a coaxial line into a half-space in terms of BIEs. In Part 2, we showed that the problem can be reduced to solving a single, scalar integral equation. Here, we convert the scalar integral equation into an infinite system of linear algebraic equations. We also express the coefficients of the system in terms of single integrals, and proceed to show how to compute them.

1. INTRODUCTION

We proceed with the numerical solution of the integral equations obtained in Part 2. The geometry is as in Figure 1.1. In Part 2, we found that the coefficients of the current density expansion can be determined by solving a single, scalar integral equation. In Section 2, we begin with the scalar integral equation, substitute in it the expansions for the unknown linear current densities and perform the two–dimensional integrations to end up with one equation in an infinite number of unknowns. The coefficients of the equation are Sommerfeld-type integrals. In Section 3, we use the orthogonality properties of the expansion functions to generate an infinite system of linear algebraic equations. The coefficients of this system are integrals of four distinct forms. In Sections 4 through 7, we discuss how the integrals are computed using Mathematica® (reference 1). In Appendix A, we evaluate analytically the integrals used in Section 3.

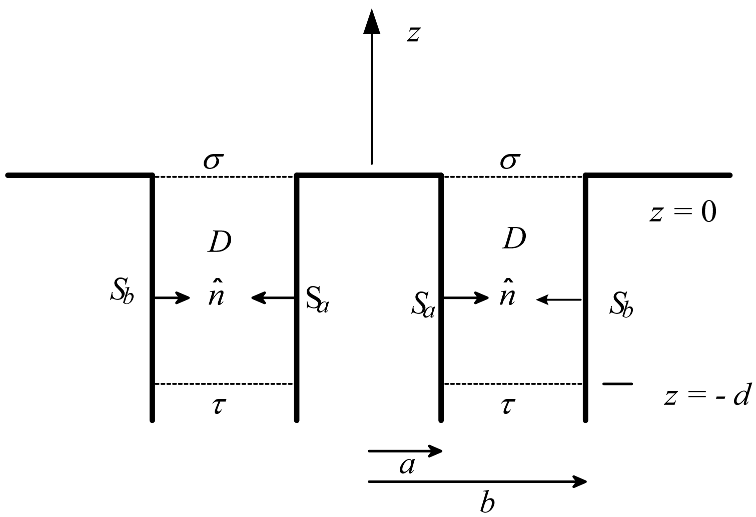

Figure 1.1. The semi-infinite coaxial line of inner radius *a* and outer radius *b* is fed at $z = -\infty$. At $z = 0$, it opens up into a half-space, its outer conductor becoming a plane that extends to infinity. All surfaces are perfectly conducting.

2. INTEGRAL EQUATION

The integral equation is given by (5.8) of Part 2 and we repeat it here

$$\int_{S_a} h_a(\mathbf{r}) \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS + \int_{S_b} h_b(\mathbf{r}) \frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial \rho'} dS = u_\sigma(\mathbf{r}') \,,\; \mathbf{r}' \in \sigma \tag{2.1}$$

where

$$u_\sigma(\mathbf{r}) = (1-A)\frac{YV}{\ln(\chi)\rho} + ikYV\sum_{n=1}^{\infty} B_n P_{0n}{}'(\rho) \tag{2.2}$$

$$h_a(\mathbf{r}) = \frac{YV}{a\ln(\chi)}\left(\mathrm{e}^{-ikz} - A\,\mathrm{e}^{+ikz}\right) - \frac{2ikYV}{\pi a}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\mathrm{e}^{i\lambda_{0n}z} \tag{2.3}$$

$$h_b(\mathbf{r}) = -\frac{YV}{b\ln(\chi)}\left(\mathrm{e}^{-ikz} - A\,\mathrm{e}^{+ikz}\right) + \frac{2ikYV}{\pi b}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}b)}\mathrm{e}^{i\lambda_{0n}z} \tag{2.4}$$

and

$$g(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\sum_{l=0}^{\infty}\varepsilon_l \cos l(\varphi-\varphi')F_l(\rho,\rho',z,z') \tag{2.5}$$

with

$$\varepsilon_0 = 1\,; \quad \varepsilon_l = 2\,, \quad l = 1,2,\ldots; \quad F_l(\rho,\rho',z,z') = \int_0^\infty J_l(\alpha\rho)J_l(\alpha\rho')\,\mathrm{e}^{-|z-z'|\sqrt{\alpha^2-k^2}}\,\frac{\alpha\, d\alpha}{\sqrt{\alpha^2-k^2}}\,. \tag{2.6}$$

We want to accelerate the convergence of the integral in (2.6). To this end, we split the integral in two parts, a static part and the rest. From (2.6), the integral corresponding to the static Green’s function is

$$H_l(\rho,\rho',z,z') = \int_0^\infty J_l(\alpha\rho)J_l(\alpha\rho')\,\mathrm{e}^{-|z-z'|\alpha}\,d\alpha \tag{2.7}$$

and what remains in (2.6) is

$$\begin{aligned} G_l(\rho,\rho',z,z') &= F_l(\rho,\rho',z,z') - H_l(\rho,\rho',z,z') \\ &= \int_0^\infty J_l(\alpha\rho)J_l(\alpha\rho')\left[\frac{\alpha\,\mathrm{e}^{-|z-z'|\sqrt{\alpha^2-k^2}}}{\sqrt{\alpha^2-k^2}} - \mathrm{e}^{-|z-z'|\alpha}\right]d\alpha\,. \end{aligned} \tag{2.8}$$

For the integral equation in (2.1), we can then write

$$
\begin{aligned}
&-\frac{1}{4\pi}\sum_{l=0}^{\infty}\varepsilon_l\int_{S_a} h_a(\mathbf{r})\cos l(\varphi-\varphi')\frac{\partial G_l(a,\rho',z,0)}{\partial\rho'}dS \\
&-\frac{1}{4\pi}\sum_{l=0}^{\infty}\varepsilon_l\int_{S_a} h_a(\mathbf{r})\cos l(\varphi-\varphi')\frac{\partial H_l(a,\rho',z,0)}{\partial\rho'}dS \\
&-\frac{1}{4\pi}\sum_{l=0}^{\infty}\varepsilon_l\int_{S_b} h_b(\mathbf{r})\cos l(\varphi-\varphi')\frac{\partial G_l(b,\rho',z,0)}{\partial\rho'}dS \\
&-\frac{1}{4\pi}\sum_{l=0}^{\infty}\varepsilon_l\int_{S_b} h_b(\mathbf{r})\cos l(\varphi-\varphi')\frac{\partial H_l(b,\rho',z,0)}{\partial\rho'}dS = u_\sigma(\mathbf{r}')\,,\ \mathbf{r}'\in\sigma\,.
\end{aligned}
\tag{2.9}
$$

From (2.2), (2.3), and (2.4), the current densities are independent of the angular direction. We can then replace (2.9) by

$$
\begin{aligned}
&-\frac{a}{2}\int_{-\infty}^{0} h_a(\mathbf{r})\frac{\partial G_0(a,\rho',z,0)}{\partial\rho'}dz-\frac{a}{2}\int_{-\infty}^{0} h_a(\mathbf{r})\frac{\partial H_0(a,\rho',z,0)}{\partial\rho'}dz \\
&-\frac{b}{2}\int_{-\infty}^{0} h_b(\mathbf{r})\frac{\partial G_0(b,\rho',z,0)}{\partial\rho'}dz-\frac{b}{2}\int_{-\infty}^{0} h_b(\mathbf{r})\frac{\partial H_0(b,\rho',z,0)}{\partial\rho'}dz = u_\sigma(\mathbf{r}')\,,\ \mathbf{r}'\in\sigma\,.
\end{aligned}
\tag{2.10}
$$

We deal with each of these integrals separately. For the second one, we use (2.3) and (2.7) to write

$$
\begin{aligned}
I_2 &= \frac{a}{2}\int_{-\infty}^{0} h_a(\mathbf{r})\frac{\partial H_0(a,\rho',z,0)}{\partial\rho'}dz = \frac{YV}{2\ln(\chi)}\int_{-\infty}^{0}\left(\mathrm{e}^{-ikz}-A\,\mathrm{e}^{+ikz}\right)\frac{\partial H_0(a,\rho',z,0)}{\partial\rho'}dz \\
&-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\int_{-\infty}^{0}\mathrm{e}^{i\lambda_{0n}z}\,\frac{\partial H_0(a,\rho',z,0)}{\partial\rho'}dz \\
&= \frac{YV}{2\ln(\chi)}\int_{0}^{\infty}dz\left(\mathrm{e}^{ikz}-A\,\mathrm{e}^{-ikz}\right)\frac{\partial}{\partial\rho'}\int_{0}^{\infty}d\alpha J_0(\alpha a)J_0(\alpha\rho')\,\mathrm{e}^{-z\alpha} \\
&-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\int_{0}^{\infty}dz\,\mathrm{e}^{-i\lambda_{0n}z}\,\frac{\partial}{\partial\rho'}\int_{0}^{\infty}d\alpha J_0(\alpha a)J_0(\alpha\rho')\,\mathrm{e}^{-z\alpha}\,.
\end{aligned}
\tag{2.11}
$$

We require that $z\ge\varepsilon>0$ so as to make the integrals in α converge absolutely and uniformly. This allows for interchanging operations; thus,

$$
\begin{aligned}
I_2 &= \frac{YV}{2\ln(\chi)}\lim_{\varepsilon\to0}\frac{\partial}{\partial\rho'}\int_{0}^{\infty}d\alpha J_0(\alpha a)J_0(\alpha\rho')\int_{\varepsilon}^{\infty}dz\left(\mathrm{e}^{ikz}-A\,\mathrm{e}^{-ikz}\right)\mathrm{e}^{-z\alpha} \\
&-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\lim_{\varepsilon\to0}\frac{\partial}{\partial\rho'}\int_{0}^{\infty}d\alpha J_0(\alpha a)J_0(\alpha\rho')\int_{\varepsilon}^{\infty}dz\,\mathrm{e}^{-i\lambda_{0n}z}\,\mathrm{e}^{-z\alpha} \\
&= \frac{YV}{2\ln(\chi)}\lim_{\varepsilon\to0}\frac{\partial}{\partial\rho'}\int_{0}^{\infty}d\alpha J_0(\alpha a)J_0(\alpha\rho')\left[\frac{-\mathrm{e}^{(ik-\alpha)\varepsilon}}{ik-\alpha}-A\frac{\mathrm{e}^{-(ik+\alpha)\varepsilon}}{ik+\alpha}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\lim_{\varepsilon\to 0}\frac{\partial}{\partial\rho'}\int_0^{\infty}d\alpha J_0(\alpha a)J_0(\alpha\rho')\frac{\mathrm{e}^{-(i\lambda_{0n}+\alpha)\varepsilon}}{i\lambda_{0n}+\alpha}\\
&=\frac{YV}{2\ln(\chi)}\lim_{\varepsilon\to 0}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\alpha\left[\frac{\mathrm{e}^{(ik-\alpha)\varepsilon}}{ik-\alpha}+A\frac{\mathrm{e}^{-(ik+\alpha)\varepsilon}}{ik+\alpha}\right]\\
&+\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\lim_{\varepsilon\to 0}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\alpha\frac{\mathrm{e}^{-(i\lambda_{0n}+\alpha)\varepsilon}}{i\lambda_{0n}+\alpha}\,.
\end{aligned}
\tag{2.12}
$$

We re-write this as

$$
\begin{aligned}
I_2&=\frac{YV}{2\ln(\chi)}\lim_{\varepsilon\to 0}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\left[-\mathrm{e}^{(ik-\alpha)\varepsilon}+A\,\mathrm{e}^{-(ik+\alpha)\varepsilon}\right]\\
&-\frac{ikYV}{2\ln(\chi)}\lim_{\varepsilon\to 0}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\left[\frac{\mathrm{e}^{(ik-\alpha)\varepsilon}}{\alpha-ik}+A\frac{\mathrm{e}^{-(ik+\alpha)\varepsilon}}{\alpha+ik}\right]\\
&+\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\lim_{\varepsilon\to 0}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\,\mathrm{e}^{-(i\lambda_{0n}+\alpha)\varepsilon}\\
&-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{i\lambda_{0n}B_n}{Y_0(\nu_{0n}a)}\lim_{\varepsilon\to 0}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\frac{\mathrm{e}^{-(i\lambda_{0n}+\alpha)\varepsilon}}{i\lambda_{0n}+\alpha}\,.
\end{aligned}
\tag{2.13}
$$

In the second and fourth terms, the limits can be moved inside the integrals. The first and third are, by definition, Weber-Schafheitlin discontinuous integrals (reference 2, pp. 398-410). Dropping the limit notation, and thus having only one such integral, we have that (reference 2, p. 404 and reference 3, p. 100)

$$
\int_0^{\infty}J_{\mu}(u\alpha)J_{\mu+1}(v\alpha)d\alpha=\begin{cases}u^{\mu}v^{-(\mu+1)}, & 0<u<v\\[4pt] \dfrac{1}{2u}, & u=v\\[4pt] 0, & u>v\end{cases}\,.
\tag{2.14}
$$

We then have

$$
\begin{aligned}
I_2&=\frac{YV(A-1)}{2\ln(\chi)\rho'}-\frac{ikYV}{2\ln(\chi)}\int_0^{\infty}d\alpha J_0(\alpha a)J_1(\alpha\rho')\left[\frac{1}{\alpha-ik}+\frac{A}{\alpha+ik}\right]\\
&+\frac{ikYV}{\pi\rho'}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{i\lambda_{0n}B_n}{Y_0(\nu_{0n}a)}\int_0^{\infty}J_0(\alpha a)J_1(\alpha\rho')\frac{d\alpha}{i\lambda_{0n}+\alpha}\,.
\end{aligned}
\tag{2.15}
$$

The fourth integral in (2.10) is the same as the second with a replaced by b and a reversal in sign because of (2.3) and (2.4); thus, we have

$$I_4 = \frac{b}{2}\int_{-\infty}^{0} h_b(\mathbf{r})\frac{\partial H_0(b,\rho',z,0)}{\partial\rho'}dz = \frac{ikYV}{2\ln(\chi)}\int_0^\infty d\alpha J_0(\alpha b)J_1(\alpha\rho')\left[\frac{1}{\alpha-ik}+\frac{A}{\alpha+ik}\right]$$
$$+\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{i\lambda_{0n}B_n}{Y_0(\nu_{0n}b)}\int_0^\infty J_0(\alpha b)J_1(\alpha\rho')\frac{d\alpha}{i\lambda_{0n}+\alpha}. \qquad (2.16)$$

We turn to the first integral in (2.10)

$$I_1 = \frac{a}{2}\int_{-\infty}^{0} h_a(\mathbf{r})\frac{\partial G_0(a,\rho',z,0)}{\partial\rho'}dz$$
$$= \frac{a}{2}\int_{-\infty}^{0} dz h_a(\mathbf{r})\frac{\partial}{\partial\rho'}\int_0^\infty J_0(\alpha a)J_0(\alpha\rho')\left[\frac{\alpha\,\mathrm{e}^{z\sqrt{\alpha^2-k^2}}}{\sqrt{\alpha^2-k^2}}-\mathrm{e}^{z\alpha}\right]d\alpha$$
$$= \frac{YV}{2\ln(\chi)}\int_{-\infty}^{0} dz\left(\mathrm{e}^{-ikz}-A\,\mathrm{e}^{+ikz}\right)\frac{\partial}{\partial\rho'}\int_0^\infty J_0(\alpha a)J_0(\alpha\rho')\left[\frac{\alpha\,\mathrm{e}^{z\sqrt{\alpha^2-k^2}}}{\sqrt{\alpha^2-k^2}}-\mathrm{e}^{z\alpha}\right]d\alpha$$
$$-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\int_{-\infty}^{0} dz\,\mathrm{e}^{i\lambda_{0n}z}\frac{\partial}{\partial\rho'}\int_0^\infty J_0(\alpha a)J_0(\alpha\rho')\left[\frac{\alpha\,\mathrm{e}^{z\sqrt{\alpha^2-k^2}}}{\sqrt{\alpha^2-k^2}}-\mathrm{e}^{z\alpha}\right]d\alpha. \qquad (2.16)$$

We re-write this as

$$I_1 = \frac{YV}{2\ln(\chi)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha J_0(\alpha a)J_0(\alpha\rho')\int_0^\infty dz\left(\mathrm{e}^{ikz}-A\,\mathrm{e}^{-ikz}\right)\left[\frac{\alpha\,\mathrm{e}^{-z\sqrt{\alpha^2-k^2}}}{\sqrt{\alpha^2-k^2}}-\mathrm{e}^{-z\alpha}\right]$$
$$-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha J_0(\alpha a)J_0(\alpha\rho')\int_0^\infty dz\,\mathrm{e}^{-i\lambda_{0n}z}\left[\frac{\alpha\,\mathrm{e}^{-z\sqrt{\alpha^2-k^2}}}{\sqrt{\alpha^2-k^2}}-\mathrm{e}^{-z\alpha}\right]$$
$$= \frac{YV}{2\ln(\chi)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha J_0(\alpha a)J_0(\alpha\rho')\left[\frac{-\alpha}{\sqrt{\alpha^2-k^2}}\left(\frac{1}{ik-\sqrt{\alpha^2-k^2}}+\frac{A}{ik+\sqrt{\alpha^2-k^2}}\right)+\frac{1}{ik-\alpha}+\frac{A}{ik+\alpha}\right]$$
$$-\frac{ikYV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha J_0(\alpha a)J_0(\alpha\rho')\left[\frac{\alpha}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)}-\frac{1}{i\lambda_{0n}+\alpha}\right]. \qquad (2.18)$$

We compute

$$-\frac{\alpha}{\sqrt{\alpha^2-k^2}\left(ik-\sqrt{\alpha^2-k^2}\right)}+\frac{1}{ik-\alpha} = \frac{\sqrt{\alpha^2-k^2}\left(ik-\sqrt{\alpha^2-k^2}\right)-\alpha(ik-\alpha)}{\sqrt{\alpha^2-k^2}\left(ik-\sqrt{\alpha^2-k^2}\right)(ik-\alpha)}$$

$$= \frac{ik\left(\sqrt{\alpha^2-k^2}-\alpha\right)+k^2}{\sqrt{\alpha^2-k^2}\left(ik-\sqrt{\alpha^2-k^2}\right)\left(ik-\alpha\right)} = \frac{-ik^3+k^2\left(\sqrt{\alpha^2-k^2}+\alpha\right)}{\sqrt{\alpha^2-k^2}\left(ik-\sqrt{\alpha^2-k^2}\right)\left(ik-\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}$$

$$= k^2\frac{-ik+\sqrt{\alpha^2-k^2}+\alpha}{\sqrt{\alpha^2-k^2}\left(ik-\sqrt{\alpha^2-k^2}\right)\left(ik-\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}$$

$$= -k^2\frac{\alpha^2+\alpha\left(ik+\sqrt{\alpha^2-k^2}\right)}{\alpha^2\sqrt{\alpha^2-k^2}\left(ik-\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)} = -k^2\frac{\alpha+ik+\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(ik-\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)} \tag{2.19}$$

$$\frac{-\alpha}{\sqrt{\alpha^2-k^2}\left(ik+\sqrt{\alpha^2-k^2}\right)}+\frac{1}{ik+\alpha} = -k^2\frac{\alpha-ik+\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(ik+\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)} \tag{2.20}$$

$$\frac{\alpha}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)}-\frac{1}{i\lambda_{0n}+\alpha} = \frac{\alpha\left(i\lambda_{0n}+\alpha\right)-\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)}$$

$$= \frac{i\lambda_{0n}\left(\alpha-\sqrt{\alpha^2-k^2}\right)+k^2}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)}$$

$$= k^2\frac{i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)} \tag{2.21}$$

We substitute the last four expressions in (2.18) to get

$$I_1 = -\frac{YVk^2}{2\ln\left(\chi\right)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha J_0(\alpha a)J_0(\alpha\rho')\left[\frac{\alpha+ik+\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(ik-\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\right.$$

$$\left.+A\frac{\alpha-ik+\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(ik+\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\right]$$

$$-\frac{ik^3YV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}a)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\frac{J_0(\alpha a)J_0(\alpha\rho')\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}. \tag{2.22}$$

Similarly, for the third integral in (2.10), we have

$$
\begin{aligned}
I_3 &= \frac{b}{2}\int_{-\infty}^{0} h_b(\mathbf{r})\frac{\partial G_0(b,\rho',z,0)}{\partial\rho'}dz \\
&= \frac{YVk^2}{2\ln\left(\chi\right)}\frac{\partial}{\partial\rho'}\int_0^{\infty} d\alpha J_0(\alpha b)J_0(\alpha\rho')\left[\frac{\alpha+ik+\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(ik-\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\right. \\
&\left.+A\left[\frac{\alpha-ik+\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(ik+\alpha\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\right]\right\} \\
&+\frac{ik^3YV}{\pi}\sum_{n=1}^{\infty}\frac{B_n}{Y_0(\nu_{0n}b)}\frac{\partial}{\partial\rho'}\int_0^{\infty} d\alpha\,\frac{J_0(\alpha b)J_0(\alpha\rho')\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}.
\end{aligned}
\tag{2.23}
$$

In the next section, we convert (2.10) to an infinite system of linear algebraic equations.

3. SYSTEM OF EQUATIONS

We write (2.10) in the form

$$-\left(I_1+I_2+I_3+I_4\right)=u_\sigma(\mathbf{r}'),\ \mathbf{r}'\in\sigma \tag{3.1}$$

where the four terms on the left are given by (2.15), (2.16), (2.22), and (2.23), while the term on the right is given by (2.2). We proceed to collect coefficients of the unknown terms

$$\begin{aligned}
&A\left\{\frac{YVk^2}{2\ln(\chi)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\,\frac{J_0(\alpha a)J_0(\alpha\rho')\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(ik+\alpha)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\right.\\
&-\frac{YVk^2}{2\ln(\chi)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\,\frac{J_0(\alpha b)J_0(\alpha\rho')\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(ik+\alpha)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}+\frac{YV}{2\ln(\chi)\rho'}\\
&\left.+\frac{ikYV}{2\ln(\chi)}\int_0^\infty d\alpha\,\frac{J_0(\alpha a)J_1(\alpha\rho')}{\alpha+ik}-\frac{ikYV}{2\ln(\chi)}\int_0^\infty d\alpha\,\frac{J_0(\alpha b)J_1(\alpha\rho')}{\alpha+ik}\right\}\\
&+\frac{ik^3YV}{\pi}\sum_{n=1}^\infty B_n\left\{\frac{1}{Y_0(\nu_{0n}a)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\,\frac{J_0(\alpha a)J_0(\alpha\rho')\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)(i\lambda_{0n}+\alpha)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}\right.\\
&-\frac{1}{Y_0(\nu_{0n}b)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\,\frac{J_0(\alpha b)J_0(\alpha\rho')\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)(i\lambda_{0n}+\alpha)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}-\frac{1}{k^2\rho'}\frac{1}{Y_0(\nu_{0n}a)}\\
&\left.+\frac{i\lambda_{0n}}{k^2Y_0(\nu_{0n}a)}\int_0^\infty d\alpha\,\frac{J_0(\alpha a)J_1(\alpha\rho')}{i\lambda_{0n}+\alpha}-\frac{i\lambda_{0n}}{k^2Y_0(\nu_{0n}b)}\int_0^\infty J_0(\alpha b)J_1(\alpha\rho')\frac{d\alpha}{i\lambda_{0n}+\alpha}-\frac{\pi}{k^2}P_{0n}{}'(\rho)\right\}\\
&=-\frac{YVk^2}{2\ln(\chi)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\,\frac{J_0(\alpha a)J_0(\alpha\rho')\left(\alpha+ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(ik-\alpha)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\\
&+\frac{YVk^2}{2\ln(\chi)}\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\,\frac{J_0(\alpha b)J_0(\alpha\rho')\left(\alpha+ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(ik-\alpha)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\\
&+\frac{YV}{2\ln(\chi)\rho'}-\frac{ikYV}{2\ln(\chi)}\int_0^\infty d\alpha\,\frac{J_0(\alpha a)J_1(\alpha\rho')}{\alpha-ik}+\frac{ikYV}{2\ln(\chi)}\int_0^\infty d\alpha\,\frac{J_0(\alpha b)J_1(\alpha\rho')}{\alpha-ik}\,.
\end{aligned}\tag{3.2}$$

We combine terms above to get

$$\frac{A}{2\ln\left(\chi\right)}\left\{k^2\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]J_0(\alpha\rho')\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(ik+\alpha)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}+\frac{1}{\rho'}\right.$$
$$\left.+ik\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]J_1(\alpha\rho')}{\alpha+ik}\right\}$$
$$+\frac{ik^3}{\pi}\sum_{n=1}^{\infty}B_n\left\{\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\frac{\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]J_0(\alpha\rho')\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}\right.$$
$$\left.-\frac{1}{k^2Y_0(\nu_{0n}a)\rho'}+\frac{i\lambda_{0n}}{k^2}\int_0^\infty d\alpha\frac{\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]J_1(\alpha\rho')}{i\lambda_{0n}+\alpha}-\frac{\pi}{k^2}P_{0n}{}'(\rho)\right\}$$
$$=\frac{1}{2\ln\left(\chi\right)}\left\{-k^2\frac{\partial}{\partial\rho'}\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]J_0(\alpha\rho')\left(\alpha+ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(ik-\alpha)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}\right.$$
$$\left.+\frac{1}{\rho'}-ik\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]J_1(\alpha\rho')}{\alpha-ik}\right\}. \tag{3.3}$$

To convert the integral equation to a system of linear algebraic equations, we first integrate both sides of this with respect to ρ' from *a* to *b*. In effect, we perform the integration

$$\int_a^b J_1(\alpha\rho)d\rho=\frac{1}{\alpha}\int_{\alpha a}^{\alpha b}J_1(t)dt=-\frac{1}{\alpha}\int_{\alpha a}^{\alpha b}J_0{}'(t)dt=\frac{J_0(\alpha a)-J_0(\alpha b)}{\alpha} \tag{3.4}$$

and also use the results in Appendix A. The end result is

$$\frac{A}{2\ln\left(\chi\right)}\left\{-k^2\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(\alpha+ik)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}+\ln\left(\chi\right)+ik\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2}{\alpha\left(\alpha+ik\right)}\right\}$$
$$+\frac{ik^3}{\pi}\sum_{n=1}^{\infty}B_n\left\{-\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}\right.$$

$$-\frac{\ln(\chi)}{k^2 Y_0(\nu_{0n}a)}+\frac{i\lambda_{0n}}{k^2}\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]}{\alpha\left(i\lambda_{0n}+\alpha\right)}\Bigg\}$$

$$=\frac{1}{2\ln(\chi)}\Bigg\{-k^2\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2\left(\alpha+ik+\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\,(\alpha-ik)\left(\sqrt{\alpha^2-k^2}+\alpha\right)}$$

$$+\ln(\chi)-ik\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2}{\alpha(\alpha-ik)}\Bigg\}. \tag{3.5}$$

We next multiply (3.2) by $\rho' P_{0m}{}'(\rho')$ and integrate from *a* to *b*

$$\frac{A}{\pi\ln(\chi)}\Bigg\{k^2\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\,(\alpha+ik)\left(\sqrt{\alpha^2-k^2}+\alpha\right)\left(\alpha^2-\nu_{0m}{}^2\right)}\alpha$$

$$-ik\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{(\alpha+ik)\left(\alpha^2-\nu_{0m}{}^2\right)}\alpha\Bigg\}$$

$$+\frac{i2k^3}{\pi^2}\sum_{n=1}^\infty B_n\Bigg\{\int_0^\infty d\alpha\frac{\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)\alpha^2}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)(i\lambda_{0n}+\alpha)\left(\alpha+\sqrt{\alpha^2-k^2}\right)\left(\alpha^2-\nu_{0m}{}^2\right)}$$

$$-\frac{i\lambda_{0n}}{k^2}\int_0^\infty d\alpha\frac{\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]\alpha}{(i\lambda_{0n}+\alpha)\left(\alpha^2-\nu_{0m}{}^2\right)}-\frac{1}{k^2}\left\{\frac{1}{\left[Y_0(\nu_{0n}b)\right]^2}-\frac{1}{\left[Y_0(\nu_{0n}a)\right]^2}\right\}\delta_{nm}\Bigg\}$$

$$=\frac{1}{\pi\ln(\chi)}\Bigg\{k^2\int_0^\infty d\alpha\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]\left(\alpha+ik+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\,(\alpha-ik)\left(\sqrt{\alpha^2-k^2}+\alpha\right)\left(\alpha^2-\nu_{0m}{}^2\right)}\alpha$$

$$\left. + ik\int_0^\infty d\alpha \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)} - \dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{\left(\alpha - ik\right)\left(\alpha^2 - \nu_{0m}{}^2\right)}\alpha \right\}, \quad m = 1, 2, \dots . \tag{3.6}$$

The last two expressions constitute an infinite system of linear algebraic equations in an infinite number of unknowns (A and the B_n). If in (3.5) we set all the B_n equal to zero, then we get for A

$$A = \frac{\ln\left(\chi\right) - ik\int_0^\infty d\alpha \dfrac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\left(\alpha - ik\right)} - k^2\int_0^\infty d\alpha \dfrac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2\left(\alpha + ik + \sqrt{\alpha^2 - k^2}\right)}{\alpha\sqrt{\alpha^2 - k^2}\left(\alpha - ik\right)\left(\sqrt{\alpha^2 - k^2} + \alpha\right)}}{\ln\left(\chi\right) + ik\int_0^\infty d\alpha \dfrac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\left(\alpha + ik\right)} - k^2\int_0^\infty d\alpha \dfrac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2\left(\alpha - ik + \sqrt{\alpha^2 - k^2}\right)}{\alpha\sqrt{\alpha^2 - k^2}\left(\alpha + ik\right)\left(\sqrt{\alpha^2 - k^2} + \alpha\right)}} \tag{3.7}$$

We can combine the integrals to reduce this to

$$A = \frac{\ln\left(\chi\right) - ik\int_0^\infty d\alpha \dfrac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2 - k^2}}}{\ln\left(\chi\right) + ik\int_0^\infty d\alpha \dfrac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2 - k^2}}} . \tag{3.8}$$

The issue we must address now is whether we should use such combinations in (3.5) and (3.6). What we have accomplished in splitting up the Green's function is to increase the rate of convergence of the integrals in α by an order of magnitude. Had we not split the Green's function, the integrals in (3.8) would behave as α^{-2}, as $\alpha \to \infty$. In (3.7) and (3.8), the integrands behave as α^{-3}. We have thus gained an order of magnitude in the rate of convergence. In fact, in the integral that does not involve the static term, the behavior is α^{-4}. In the static term, we altered the original behavior of α^{-2} by integrating part of the expression, with the rest behaving as α^{-3}. This is true of all four terms. Since the acceleration of convergence has been accomplished, we should feel free to recombine the remaining terms. The only reservation here is the following. Looking at the numerator of (3.7), we see that the first integrals behaves as α^{-3}, while the second can be split into two integrals, one behaving as α^{-4} and the other as α^{-5}. Whether the last two terms are much smaller than the first cannot be concluded from these observations. We proceed then to combine terms in (3.5) and (3.6).

First, we perform the following calculations

$$\frac{i\lambda_{0n}}{k^2\alpha\left(i\lambda_{0n}+\alpha\right)}-\frac{i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{i\lambda_{0n}\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)-k^2\alpha\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{i\lambda_{0n}\sqrt{\alpha^2-k^2}\left[\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\alpha+i\lambda_{0n}\sqrt{\alpha^2-k^2}+\alpha^2-k^2\right]-k^2\alpha\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{i\lambda_{0n}\sqrt{\alpha^2-k^2}\left[\left(\alpha+i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\alpha+i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right]-k^2\alpha\left(i\lambda_{0n}+\alpha+\sqrt{\alpha^2-k^2}\right)}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{\left(\alpha+i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\alpha\left[i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right]+i\lambda_{0n}\sqrt{\alpha^2-k^2}\left[i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right]}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{\left[i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right]\left[\left(\alpha+i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\alpha+i\lambda_{0n}\sqrt{\alpha^2-k^2}\right]}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{\left[i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right]\left[\left(\alpha+\sqrt{\alpha^2-k^2}\right)\alpha+i\lambda_{0n}\left(\alpha+\sqrt{\alpha^2-k^2}\right)\right]}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(i\lambda_{0n}+\alpha\right)\left(\alpha+\sqrt{\alpha^2-k^2}\right)}$$

$$=\frac{i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2}{k^2\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)}. \tag{3.9}$$

Also,

$$\frac{k^2\alpha\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)}{\sqrt{\alpha^2-k^2}\left(\alpha+ik\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)\left(\alpha^2-{\nu_{0m}}^2\right)}-\frac{ik\alpha}{\left(\alpha+ik\right)\left(\alpha^2-{\nu_{0m}}^2\right)}$$

$$=\frac{k^2\alpha\left(\alpha-ik+\sqrt{\alpha^2-k^2}\right)-ik\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\alpha^2-k^2}+\alpha\right)}{\sqrt{\alpha^2-k^2}\left(\alpha+ik\right)\left(\sqrt{\alpha^2-k^2}+\alpha\right)\left(\alpha^2-{\nu_{0m}}^2\right)}$$

$$= -ik\alpha \frac{ik\left(\alpha - ik + \sqrt{\alpha^2 - k^2}\right) + \alpha^2 - k^2 + \alpha\sqrt{\alpha^2 - k^2}}{\sqrt{\alpha^2 - k^2}\,(\alpha + ik)\left(\sqrt{\alpha^2 - k^2} + \alpha\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= -ik\alpha \frac{ik\left(\alpha + \sqrt{\alpha^2 - k^2}\right) + \alpha^2 + \alpha\sqrt{\alpha^2 - k^2}}{\sqrt{\alpha^2 - k^2}\,(\alpha + ik)\left(\sqrt{\alpha^2 - k^2} + \alpha\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= -ik\alpha \frac{\left(\alpha + \sqrt{\alpha^2 - k^2}\right)(ik + \alpha)}{\sqrt{\alpha^2 - k^2}\,(\alpha + ik)\left(\sqrt{\alpha^2 - k^2} + \alpha\right)\left(\alpha^2 - {\nu_{0m}}^2\right)} = \frac{-ik\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} \tag{3.10}$$

while

$$\frac{\left(i\lambda_{0n} + \alpha + \sqrt{\alpha^2 - k^2}\right)\alpha^2}{\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)} - \frac{i\lambda_{0n}\alpha}{k^2\,(i\lambda_{0n} + \alpha)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= \alpha \frac{\left(i\lambda_{0n} + \alpha + \sqrt{\alpha^2 - k^2}\right)\alpha k^2 - i\lambda_{0n}\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= \alpha \frac{\left(i\lambda_{0n} + \alpha + \sqrt{\alpha^2 - k^2}\right)\alpha k^2 - i\lambda_{0n}\sqrt{\alpha^2 - k^2}\left[i\lambda_{0n}\left(\alpha + \sqrt{\alpha^2 - k^2}\right) + \alpha\sqrt{\alpha^2 - k^2} + \alpha^2 - k^2\right]}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= \alpha \frac{\left(i\lambda_{0n} + \alpha + \sqrt{\alpha^2 - k^2}\right)\alpha k^2 - i\lambda_{0n}\sqrt{\alpha^2 - k^2}\left[\alpha\left(\alpha + i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right) + i\lambda_{0n}\sqrt{\alpha^2 - k^2} - k^2\right]}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= \alpha \frac{\left(\alpha + i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)\left(k^2 - i\lambda_{0n}\sqrt{\alpha^2 - k^2}\right)\alpha - i\lambda_{0n}\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n}\sqrt{\alpha^2 - k^2} - k^2\right)}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= -\frac{\alpha\left(i\lambda_{0n}\sqrt{\alpha^2 - k^2} - k^2\right)\left[\left(\alpha + i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)\alpha + i\lambda_{0n}\sqrt{\alpha^2 - k^2}\right]}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= -\frac{\alpha\left(i\lambda_{0n}\sqrt{\alpha^2 - k^2} - k^2\right)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)(\alpha + i\lambda_{0n})}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)(i\lambda_{0n} + \alpha)\left(\alpha + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= -\frac{\alpha\left(i\lambda_{0n}\sqrt{\alpha^2 - k^2} - k^2\right)}{k^2\sqrt{\alpha^2 - k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2 - k^2}\right)\left(\alpha^2 - {\nu_{0m}}^2\right)} \tag{3.11}$$

and

$$\frac{k^2\left(\alpha+ik+\sqrt{\alpha^2-k^2}\right)\alpha}{\sqrt{\alpha^2-k^2}\,(\alpha-ik)\left(\sqrt{\alpha^2-k^2}+\alpha\right)\left(\alpha^2-{\nu_{0m}}^2\right)}+\frac{ik\alpha}{(\alpha-ik)\left(\alpha^2-{\nu_{0m}}^2\right)}=\frac{ik\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-{\nu_{0m}}^2\right)}. \tag{3.12}$$

We substitute these results in (3.5) and (3.6). From (3.5)

$$\begin{aligned}
&\frac{A}{2\ln(\chi)}\left\{\ln(\chi)+ik\int_0^\infty d\alpha\,\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2-k^2}}\right\}\\
&+\frac{ik}{\pi}\sum_{n=1}^\infty B_n\left\{-\frac{\ln(\chi)}{Y_0(\nu_{0n}a)}\right.\\
&\qquad\left.+\int_0^\infty d\alpha\,\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\dfrac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left(i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)}\right\}\\
&=\frac{1}{2\ln(\chi)}\left\{\ln(\chi)-ik\int_0^\infty d\alpha\,\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2-k^2}}\right\}
\end{aligned} \tag{3.13}$$

while from (3.6)

$$\begin{aligned}
&-A\left\{\int_0^\infty d\alpha\,\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{\sqrt{\alpha^2-k^2}\left(\alpha^2-{\nu_{0m}}^2\right)}\alpha\right\}\\
&-\frac{2\ln(\chi)}{\pi}\sum_{n=1}^\infty B_n\left\{\left[\frac{1}{\left[Y_0(\nu_{0n}b)\right]^2}-\frac{1}{\left[Y_0(\nu_{0n}a)\right]^2}\right]\delta_{nm}\right.\\
&\left.+\int_0^\infty d\alpha\,\frac{\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\dfrac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]\left(i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)\left(\alpha^2-{\nu_{0m}}^2\right)}\right\}\\
&=\int_0^\infty d\alpha\,\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{\sqrt{\alpha^2-k^2}\left(\alpha^2-{\nu_{0m}}^2\right)}\alpha\,,\quad m=1,2,\ldots.
\end{aligned} \tag{3.14}$$

The last two expressions constitute the final system of equations.

4. CALCULATION OF THE FIRST INTEGRAL IN (3.13)

The first integral in (3.13) is

$$I = \int_0^\infty d\alpha \, \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2 - k^2}} \,. \tag{4.1}$$

The square root in this (and all other expressions) must be chosen carefully. It appeared first in the Green's function Fourier-series expansion, specifically in the integral in (2.6). For this integral to converge, the square root must be real. With $\varepsilon > 0$ but small, we replace the original expression by $\sqrt{\alpha^2 - k^2 + i\varepsilon}$. This root must have a positive real part for $\alpha > k$

$$\begin{aligned}\sqrt{\alpha^2 - k^2 + i\varepsilon} &= \left[\left(\alpha^2 - k^2\right)^2 + \varepsilon^2\right]^{\frac{1}{4}} \mathrm{e}^{i\frac{1}{2}\tan^{-1}\left(\frac{\varepsilon}{\alpha^2 - k^2}\right)} \\ &\approx \sqrt{\alpha^2 - k^2}\left[1 + \frac{i}{2}\left(\frac{\varepsilon}{\alpha^2 - k^2}\right)\right], \quad \alpha > k\end{aligned} \tag{4.2}$$

which indeed has a positive real part. For $k > \alpha$, we must choose the branch of the arctangent function that gives a positive real part and has a dominant imaginary part as $\varepsilon \to 0^+$. The appropriate branch is the one between $\pi/2$ and $3\pi/2$

$$\begin{aligned}\sqrt{\alpha^2 - k^2 + i\varepsilon} &= \left[\left(\alpha^2 - k^2\right)^2 + \varepsilon^2\right]^{\frac{1}{4}} \mathrm{e}^{i\frac{1}{2}\left[\pi - \tan^{-1}\left(\frac{\varepsilon}{k^2 - \alpha^2}\right)\right]} \\ &\approx i\sqrt{k^2 - \alpha^2}\left[1 - \frac{i}{2}\left(\frac{\varepsilon}{k^2 - \alpha^2}\right)\right] = \frac{\varepsilon}{2\sqrt{k^2 - \alpha^2}} + i\sqrt{k^2 - \alpha^2}\,, \quad k > \alpha\,.\end{aligned} \tag{4.3}$$

A simple calculation shows that this is equivalent to setting

$$\sqrt{\alpha^2 - k^2} = i\sqrt{k^2 - \alpha^2 - i\varepsilon}\,. \tag{4.4}$$

In general, we write

$$\sqrt{\alpha^2 - k^2} = \begin{cases} \sqrt{\alpha^2 - k^2}\,, & \alpha > k \\ i\sqrt{k^2 - \alpha^2}\,, & \alpha < k \end{cases}. \tag{4.5}$$

Using this information, we write for (4.1)

$$I = -i\int_0^k d\alpha \, \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{k^2 - \alpha^2}} + \int_k^\infty d\alpha \, \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2 - k^2}} = -iI_1 + I_2\,. \tag{4.6}$$

The first integral in (4.6) can be computed using a polynomial approximation of the Bessel function (reference 4, pp. 369-370). Since

$$k\frac{a+b}{2}<1 \tag{4.7}$$

we see that the argument of the Bessel function is smaller than two. We elaborate further on this. Let

$$a=\rho b \quad , \quad 0<\rho<1 \tag{4.8}$$

Then, in place of (4.7), we have

$$kb<\frac{2}{1+\rho}<2\,. \tag{4.9}$$

We also have that

$$k<\nu_{01} \tag{4.10}$$

where ν_{01} satisfies

$$J_0(\nu_{0n}a)Y_0(\nu_{0n}b)-J_0(\nu_{0n}b)Y_0(\nu_{0n}a)=0\,, \quad n=1,2,\ldots \tag{4.11}$$

Using the values of ν_{01} from reference 4, Table 9.7, p. 415, we can provide bounds for *kb* and *k*. The results are shown in Table 4.1.

Table 4.1. Bounds on *kb* and *k*.

$\rho=a/b$	ν_{01} (m^{-1})	$kb<$	$k<$ (m^{-1})	$\rho^{-1}=b/a$
0.8	12.56	1.111	12.56	1.25
0.6	4.697	1.250	4.697	1.67
0.4	2.073	1.429	2.073	2.5
0.2	0.7632	1.667	0.7632	5
0.1	0.3314	1.818	0.3314	10

An alternative way is to use Mathematica® (reference 1) to compute this and all other integrals. This is what we did in practice. We first re-write the first integral in (4.6) as the sum of two integrals

$$I_1=\frac{1}{k^2}\int_0^k\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2\sqrt{k^2-\alpha^2}}{\alpha}d\alpha+\frac{1}{k^2}\int_0^k\frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]^2\alpha}{\sqrt{k^2-\alpha^2}}d\alpha=\frac{1}{k^2}\left(I_{11}+I_{12}\right) \tag{4.12}$$

For the second integral, we use the transformation

$$t^2 = k^2 - \alpha^2 \tag{4.13}$$

to re-write the integral in the form

$$I_{12} = \int_0^k \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2 \alpha}{\sqrt{k^2 - \alpha^2}} d\alpha = \int_0^k \left[J_0(\sqrt{k^2 - t^2}\, a) - J_0(\sqrt{k^2 - t^2}\, b)\right]^2 dt\,. \tag{4.14}$$

For the second integral in (4.6) we use the following transformation (reference 5, p. 174)

$$t^2 = \alpha^2 - k^2 \tag{4.15}$$

to get

$$I_2 = \int_0^\infty \frac{\left[J_0(b\sqrt{t^2 + k^2}) - J_0(a\sqrt{t^2 + k^2})\right]^2}{t^2 + k^2} dt\,. \tag{4.16}$$

We split the interval of integration into two intervals

$$\begin{aligned} I_2 &= \int_0^L \frac{\left[J_0(b\sqrt{t^2 + k^2}) - J_0(a\sqrt{t^2 + k^2})\right]^2}{t^2 + k^2} dt + \int_L^\infty \frac{\left[J_0(b\sqrt{t^2 + k^2}) - J_0(a\sqrt{t^2 + k^2})\right]^2}{t^2 + k^2} dt \\ &= I_{21} + I_{22} \end{aligned} \tag{4.17}$$

where L is a large positive integer. In Mathematica® (reference 1), we set it equal to 10,000. The last integral in (4.16) is computed by using the asymptotic form of the Bessel functions

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \pi/4)\,, \quad |z| \to \infty\,. \tag{4.18}$$

Thus,

$$\begin{aligned} \left[J_0(b\alpha) - J_0(a\alpha)\right]^2 &= \frac{1}{\pi\alpha}\left\{\frac{1}{a} + \frac{1}{b} + \frac{\sin(2a\alpha)}{a} + \frac{\sin(2b\alpha)}{b}\right. \\ &\left. -2\frac{\cos\left[\alpha\left(b-a\right)\right] + \sin\left[\alpha\left(b+a\right)\right]}{\sqrt{ab}} + O\left(\frac{1}{\alpha}\right)\right\} \end{aligned} \tag{4.19}$$

and

$$\frac{1}{\alpha\sqrt{\alpha^2-k^2}} = \frac{1}{\alpha^2\sqrt{1-\left(\frac{k}{\alpha}\right)^2}} \approx \frac{1+\frac{1}{2}\left(\frac{k}{\alpha}\right)^2}{\alpha^2}. \tag{4.20}$$

We substitute in I_{22}

$$I_{22} = \int_L^\infty \frac{\left[J_0(b\sqrt{t^2+k^2}) - J_0(a\sqrt{t^2+k^2})\right]^2}{t^2+k^2} dt = \int_{\sqrt{L^2+k^2}}^\infty \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2-k^2}} d\alpha$$

$$\approx \frac{1}{\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\int_{\sqrt{L^2+k^2}}^\infty \frac{d\alpha}{\alpha^3} = \frac{1}{2\pi\left(L^2+k^2\right)}\left(\frac{1}{a}+\frac{1}{b}\right). \tag{4.21}$$

We substitute (4.12), (4.14), (4.17) and (4.21) in (4.6) to get

$$I = -i\int_0^k d\alpha \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{k^2-\alpha^2}} + \int_k^\infty d\alpha \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2}{\alpha\sqrt{\alpha^2-k^2}}$$

$$= -\frac{i}{k^2}\left\{\int_0^k \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]^2 \sqrt{k^2-\alpha^2}}{\alpha} d\alpha + \int_0^k \left[J_0(\sqrt{k^2-t^2}\,a) - J_0(\sqrt{k^2-t^2}\,b)\right]^2 dt\right\}$$

$$+\int_0^L \frac{\left[J_0(b\sqrt{t^2+k^2}) - J_0(a\sqrt{t^2+k^2})\right]^2}{t^2+k^2} dt + \frac{1}{2\pi\left(L^2+k^2\right)}\left(\frac{1}{a}+\frac{1}{b}\right). \tag{4.22}$$

This concludes the description of how we evaluate the first integral in (3.13).

5. CALCULATION OF THE SECOND INTEGRAL IN (3.13)

We continue the evaluation of the integrals in (3.13). The second integral in (3.13) is

$$I=\int_0^\infty d\alpha \frac{\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\dfrac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left(i\lambda_{0n}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(i\lambda_{0n}+\sqrt{\alpha^2-k^2}\right)}. \tag{5.1}$$

With

$$\lambda_{0n}=-i\sqrt{\nu_{0n}{}^2-k^2}\ , \quad \nu_{0n}{}^2>k^2 \tag{5.2}$$

and

$$f(\alpha;a,b;\nu_{0n})=\left[J_0(\alpha a)-J_0(\alpha b)\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right] \tag{5.3}$$

and (4.5), we write

$$\begin{aligned}
I&=\int_0^\infty d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\alpha^2-k^2}\right)}\\
&=\int_0^k d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(i\sqrt{\nu_{0n}{}^2-k^2}\sqrt{k^2-\alpha^2}-k^2\right)}{i\alpha\sqrt{k^2-\alpha^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+i\sqrt{k^2-\alpha^2}\right)}\\
&+\int_k^\infty d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\alpha^2-k^2}\right)}=I_1+I_2.
\end{aligned} \tag{5.4}$$

Consider

$$\begin{aligned}
\frac{i\sqrt{\nu_{0n}{}^2-k^2}\sqrt{k^2-\alpha^2}-k^2}{i\alpha\sqrt{k^2-\alpha^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+i\sqrt{k^2-\alpha^2}\right)}&=\frac{\left(i\sqrt{\nu_{0n}{}^2-k^2}\sqrt{k^2-\alpha^2}-k^2\right)\left(\sqrt{\nu_{0n}{}^2-k^2}-i\sqrt{k^2-\alpha^2}\right)}{i\alpha\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)}\\
&=\frac{i\left(\nu_{0n}{}^2-k^2\right)\sqrt{k^2-\alpha^2}-k^2\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\nu_{0n}{}^2-k^2}\left(k^2-\alpha^2\right)+ik^2\sqrt{k^2-\alpha^2}}{i\alpha\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)}
\end{aligned}$$

$$= \frac{i\nu_{0n}{}^2\sqrt{k^2-\alpha^2}-\alpha^2\sqrt{\nu_{0n}{}^2-k^2}}{i\alpha\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)} = \frac{\nu_{0n}{}^2}{\alpha\left(\nu_{0n}{}^2-\alpha^2\right)} + i\frac{\alpha\sqrt{\nu_{0n}{}^2-k^2}}{\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)} . \tag{5.5}$$

Then

$$\begin{aligned} I_1 &= \int_0^k d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(i\sqrt{\nu_{0n}{}^2-k^2}\sqrt{k^2-\alpha^2}-k^2\right)}{i\alpha\sqrt{k^2-\alpha^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+i\sqrt{k^2-\alpha^2}\right)} \\ &= \nu_{0n}{}^2\int_0^k \frac{f(\alpha;a,b;\nu_{0n})d\alpha}{\alpha\left(\nu_{0n}{}^2-\alpha^2\right)} + i\sqrt{\nu_{0n}{}^2-k^2}\int_0^k \frac{f(\alpha;a,b;\nu_{0n})\alpha d\alpha}{\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)} \\ &= \nu_{0n}{}^2 I_{11} + i\sqrt{\nu_{0n}{}^2-k^2}\, I_{12} . \end{aligned} \tag{5.6}$$

Let

$$s = \sqrt{k^2-\alpha^2} \ , \quad \alpha = \sqrt{k^2-s^2} \ , \quad sds = -\alpha d\alpha . \tag{5.7}$$

Then

$$I_{12} = \int_0^k \frac{f(\alpha;a,b;\nu_{0n})\alpha d\alpha}{\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)} = \int_0^k \frac{f(\sqrt{k^2-s^2};a,b;\nu_{0n})ds}{\nu_{0n}{}^2-k^2+s^2} \tag{5.8}$$

and (5.6) becomes

$$I_1 = \nu_{0n}{}^2\int_0^k \frac{f(\alpha;a,b;\nu_{0n})d\alpha}{\alpha\left(\nu_{0n}{}^2-\alpha^2\right)} + i\sqrt{\nu_{0n}{}^2-k^2}\int_0^k \frac{f(\sqrt{k^2-s^2};a,b;\nu_{0n})ds}{\nu_{0n}{}^2-k^2+s^2} . \tag{5.9}$$

We also write the second integral in (5.4) as the sum of two integrals

$$\begin{aligned} I_2 &= \int_k^\infty d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\alpha^2-k^2}\right)} \\ &= \int_k^K d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\alpha^2-k^2}\right)} \end{aligned}$$

$$+\int_K^\infty d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\alpha^2-k^2}\right)}=I_{21}+I_{22} \tag{5.10}$$

where K is a large number. The last integral is evaluated asymptotically, while, for the first, we use the transformation

$$t=\sqrt{\alpha^2-k^2}\ ,\quad \alpha=\sqrt{k^2+t^2}\ ,\quad tdt=\alpha d\alpha \tag{5.11}$$

to get

$$I_{21}=\int_k^K d\alpha \frac{f(\alpha;a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2}+\sqrt{\alpha^2-k^2}\right)}$$

$$=\int_0^{\sqrt{K^2-k^2}} \frac{f(\sqrt{k^2+t^2};a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\,t-k^2\right)}{\left(k^2+t^2\right)\left(\sqrt{\nu_{0n}{}^2-k^2}+t\right)}dt\ . \tag{5.12}$$

We choose K so that the upper limit of integration is a large integer L; thus,

$$L=\sqrt{K^2-k^2}\ ,\quad \text{or}\quad K=\sqrt{L^2+k^2}\ . \tag{5.13}$$

In the second integral in (5.10), we may rationalize the denominator to get

$$I_{22}=\int_{\sqrt{L^2+k^2}}^\infty \frac{f(\alpha;a,b;\nu_{0n})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)}d\alpha \tag{5.14}$$

where the lower limit of integration is large enough so that the integrand does not exhibit any singularities.

We replace (5.9), (5.12), and (5.14) in (5.4) to get

$$I=\nu_{0n}{}^2\int_0^k \frac{f(\alpha;a,b;\nu_{0n})d\alpha}{\alpha\left(\nu_{0n}{}^2-\alpha^2\right)}+i\sqrt{\nu_{0n}{}^2-k^2}\int_0^k \frac{f(\sqrt{k^2-s^2};a,b;\nu_{0n})ds}{\nu_{0n}{}^2-k^2+s^2}$$

$$+\int_0^L \frac{f(\sqrt{k^2+t^2};a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\,t-k^2\right)}{\left(k^2+t^2\right)\left(\sqrt{\nu_{0n}{}^2-k^2}+t\right)}dt$$

$$+\int_{\sqrt{L^2+k^2}}^{\infty} \frac{f(\alpha;a,b;\nu_{0n})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)}\,d\alpha = J_1+iJ_2+J_3+J_4\,. \tag{5.15}$$

The first three terms are computed numerically in Mathematica® (reference 1), while the last one is evaluated asymptotically.

The asymptotic evaluation proceeds as follows. From (5.3)

$$f(\alpha;a,b;\nu_{0n}) = \frac{1}{\pi\alpha}\left\{\frac{1}{aY_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0n}b)}+\frac{\sin(2a\alpha)}{aY_0(\nu_{0n}a)}+\frac{\sin(2b\alpha)}{bY_0(\nu_{0n}b)}\right.$$
$$\left.-\left[\frac{1}{Y_0(\nu_{0n}a)}+\frac{1}{Y_0(\nu_{0n}b)}\right]\frac{\cos\left[\alpha\left(b-a\right)\right]+\sin\left[\alpha\left(b+a\right)\right]}{\sqrt{ab}}+O\left(\alpha^{-1}\right)\right\} \tag{5.16}$$

while

$$\frac{\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}}{\alpha\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)} = \frac{\alpha\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{1-\left(\frac{k}{\alpha}\right)^2}}{\alpha^3\sqrt{1-\left(\frac{k}{\alpha}\right)^2}\left[1-\left(\frac{\nu_{0n}}{\alpha}\right)^2\right]}$$
$$\approx \alpha^{-3}\left\{\alpha\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\left[1-\frac{1}{2}\left(\frac{k}{\alpha}\right)^2\right]\right\}\left[1+\frac{1}{2}\left(\frac{k}{\alpha}\right)^2\right]\left[1+\left(\frac{\nu_{0n}}{\alpha}\right)^2\right]$$
$$\approx \alpha^{-3}\left\{\alpha\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\left[1-\frac{1}{2}\left(\frac{k}{\alpha}\right)^2\right]\right\}\left[1+\frac{1}{2}\left(\frac{k}{\alpha}\right)^2+\left(\frac{\nu_{0n}}{\alpha}\right)^2\right]$$
$$\approx \frac{\alpha\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2}{\alpha^3} = \frac{\sqrt{\nu_{0n}{}^2-k^2}}{\alpha^2}-\frac{\nu_{0n}{}^2}{\alpha^3}\,. \tag{5.17}$$

Thus,

$$\frac{f(\alpha;a,b;\nu_{0n})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)}$$
$$\approx \frac{1}{\pi\alpha}\left\{\frac{1}{aY_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0n}b)}+\frac{\sin(2a\alpha)}{aY_0(\nu_{0n}a)}+\frac{\sin(2b\alpha)}{bY_0(\nu_{0n}b)}\right.$$

$$-\left[\frac{1}{Y_0(\nu_{0n}a)}+\frac{1}{Y_0(\nu_{0n}b)}\right]\frac{\cos\left[\alpha\left(b-a\right)\right]+\sin\left[\alpha\left(b+a\right)\right]}{\sqrt{ab}}\Bigg\}\left\{\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\alpha^2}-\frac{\nu_{0n}{}^2}{\alpha^3}\right\}$$

$$=\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\pi\alpha^3}\left[\frac{1}{aY_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0n}b)}\right]. \tag{5.18}$$

From this and (5.15)

$$J_4=\int_{\sqrt{L^2+k^2}}^{\infty}\frac{f(\alpha;a,b;\nu_{0n})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)}{\alpha\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)}d\alpha$$

$$\approx\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\pi}\left[\frac{1}{aY_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0n}b)}\right]\int_{\sqrt{L^2+k^2}}^{\infty}\frac{d\alpha}{\alpha^3}$$

$$=\frac{\sqrt{\nu_{0n}{}^2-k^2}}{2\pi\left(L^2+k^2\right)}\left[\frac{1}{aY_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0n}b)}\right]. \tag{5.19}$$

This concludes the description of how we evaluate the second integral in (3.13). From (5.15) and (5.19), the final expression is

$$I\approx\nu_{0n}{}^2\int_0^k\frac{f(\alpha;a,b;\nu_{0n})d\alpha}{\alpha\left(\nu_{0n}{}^2-\alpha^2\right)}+i\sqrt{\nu_{0n}{}^2-k^2}\int_0^k\frac{f(\alpha;a,b;\nu_{0n})\alpha d\alpha}{\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)}$$

$$+\int_0^L\frac{f(\sqrt{k^2+t^2};a,b;\nu_{0n})\left(\sqrt{\nu_{0n}{}^2-k^2}\,t-k^2\right)}{\left(k^2+t^2\right)\left(\sqrt{\nu_{0n}{}^2-k^2}+t\right)}dt+\frac{\sqrt{\nu_{0n}{}^2-k^2}}{2\pi\left(L^2+k^2\right)}\left[\frac{1}{aY_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0n}b)}\right]. \tag{5.20}$$

6. THE FIRST INTEGRAL IN (3.14)

We proceed to evaluate the integrals in (3.14). The first integral in (3.14) is

$$I = \int_0^\infty \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)} - \dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} \alpha d\alpha\,. \tag{6.1}$$

We write this as

$$I = \int_0^\infty \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)} - \dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} \alpha d\alpha = \int_0^\infty \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= -i\int_0^k \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{k^2 - \alpha^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} + \int_k^\infty \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} = -iI_1 + I_2\,. \tag{6.2}$$

We use (5.7) to re-write this

$$I = \int_0^\infty \frac{\left[J_0(\alpha a) - J_0(\alpha b)\right]\left[\dfrac{J_0(\alpha a)}{Y_0(\nu_{0m}a)} - \dfrac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} \alpha d\alpha = \int_0^\infty \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$= i\int_0^k \frac{f(\sqrt{k^2 - s^2}; a, b; \nu_{0m})ds}{{\nu_{0m}}^2 - k^2 + s^2} + \int_k^\infty \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} = iI_1 + I_2 \tag{6.3}$$

The first integral is computed in Mathematica® (reference 2). For the second integral, we write

$$I_2 = \int_k^K \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} + \int_K^\infty \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)} = I_{21} + I_{22}\,. \tag{6.4}$$

Using (5.11), we re-write the first integral in the form

$$I_2 = \int_0^L \frac{f(\sqrt{k^2 + t^2}; a, b; \nu_{0m})dt}{t^2 - \left({\nu_{0m}}^2 - k^2\right)} + \int_{\sqrt{L^2 + k^2}}^\infty \frac{f(\alpha; a, b; \nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2 - k^2}\left(\alpha^2 - {\nu_{0m}}^2\right)}$$

$$\approx \int_0^{\sqrt{{\nu_{0m}}^2 - k^2} - 10^{-4}} \frac{f(\sqrt{k^2 + t^2}; a, b; \nu_{0m})dt}{t^2 - \left({\nu_{0m}}^2 - k^2\right)} + \int_{\sqrt{{\nu_{0m}}^2 - k^2} + 10^{-4}}^L \frac{f(\sqrt{k^2 + t^2}; a, b; \nu_{0m})dt}{t^2 - \left({\nu_{0m}}^2 - k^2\right)}$$

$$+\int_{\sqrt{L^2+k^2}}^{\infty} \frac{f(\alpha;a,b;\nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0m}{}^2\right)} \tag{6.5}$$

where L is a sufficiently large integer so that the last integral is free of singularities. The first two of these integrals are computed numerically in Mathematica® (reference 2) while the last is evaluated asymptotically. We note that the singularity at ν_{0m} is removable and that the integrand has a finite value there.

For $\alpha \to \infty$

$$f(\alpha;a,b;\nu_{0m}) = \frac{1}{\pi\alpha}\left\{\frac{1}{aY_0(\nu_{0m}a)} + \frac{1}{bY_0(\nu_{0m}b)} + \frac{\sin(2a\alpha)}{aY_0(\nu_{0m}a)} + \frac{\sin(2b\alpha)}{aY_0(\nu_{0m}b)} \right.$$
$$\left. -\left[\frac{1}{Y_0(\nu_{0m}a)} + \frac{1}{Y_0(\nu_{0m}b)}\right]\frac{\cos\left[\alpha\left(b-a\right)\right]+\sin\left[\alpha\left(b+a\right)\right]}{\sqrt{ab}} + O\left(\alpha^{-1}\right)\right\} \tag{6.6}$$

and

$$\frac{\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0m}{}^2\right)} = \frac{1}{\alpha^2\sqrt{1-\left(\frac{k}{\alpha}\right)^2}\left[1-\left(\frac{\nu_{0m}}{\alpha}\right)^2\right]}$$
$$\approx \frac{\left[1+\frac{1}{2}\left(\frac{k}{\alpha}\right)^2\right]\left[1+\left(\frac{\nu_{0m}}{\alpha}\right)^2\right]}{\alpha^2} \approx \frac{1+\frac{1}{2}\left(\frac{k}{\alpha}\right)^2+\left(\frac{\nu_{0m}}{\alpha}\right)^2}{\alpha^2} \tag{6.7}$$

so that

$$\frac{f(\alpha;a,b;\nu_{0m})\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0m}{}^2\right)} = \frac{1}{\pi\alpha^3}\left[\frac{1}{aY_0(\nu_{0m}a)} + \frac{1}{bY_0(\nu_{0m}b)}\right] + O\left(\alpha^{-4}\right). \tag{6.8}$$

We can then write

$$\int_{\sqrt{L^2+k^2}}^{\infty} \frac{f(\alpha;a,b;\nu_{0m})\alpha d\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0m}{}^2\right)} \approx \frac{1}{2\pi\left(L^2+k^2\right)}\left[\frac{1}{aY_0(\nu_{0m}a)} + \frac{1}{bY_0(\nu_{0m}b)}\right]. \tag{6.9}$$

From (6.3), (6.5), and (6.9), the final expression for the third integral is

$$I \approx i\int_0^k \frac{f(\sqrt{k^2 - s^2}; a, b; \nu_{0m})ds}{{\nu_{0m}}^2 - k^2 + s^2} + \int_0^{\sqrt{{\nu_{0m}}^2 - k^2} - 10^{-4}} \frac{f(\sqrt{k^2 + t^2}; a, b; \nu_{0m})dt}{t^2 - \left({\nu_{0m}}^2 - k^2\right)}$$

$$+ \int_{\sqrt{{\nu_{0m}}^2 - k^2} + 10^{-4}}^{L} \frac{f(\sqrt{k^2 + t^2}; a, b; \nu_{0m})dt}{t^2 - \left({\nu_{0m}}^2 - k^2\right)} + \frac{1}{2\pi\left(L^2 + k^2\right)}\left[\frac{1}{aY_0(\nu_{0m}a)} + \frac{1}{bY_0(\nu_{0m}b)}\right]. \tag{6.10}$$

7. THE SECOND INTEGRAL IN (3.14)

From (3.14), the fourth integral is

$$I = \int_0^\infty \frac{\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)} - \frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)}\right]\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)} - \frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right]\left(i\lambda_{0n}\sqrt{\alpha^2-k^2} - k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(i\lambda_{0n} + \sqrt{\alpha^2-k^2}\right)\left(\alpha^2 - \nu_{0m}{}^2\right)} d\alpha \,. \tag{7.1}$$

This is similar to the previous integral. Letting

$$g(\alpha;a,b;\nu_{0n}) = \frac{J_0(\alpha a)}{Y_0(\nu_{0n}a)} - \frac{J_0(\alpha b)}{Y_0(\nu_{0n}b)} \,. \tag{7.2}$$

and following Section 5, we write

$$\begin{aligned} I &= \int_0^\infty \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2} - k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2} + \sqrt{\alpha^2-k^2}\right)\left(\alpha^2-\nu_{0m}{}^2\right)} d\alpha \\ &= \int_0^k \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(i\sqrt{\nu_{0n}{}^2-k^2}\sqrt{k^2-\alpha^2} - k^2\right)\alpha}{i\sqrt{k^2-\alpha^2}\left(\sqrt{\nu_{0n}{}^2-k^2} + i\sqrt{k^2-\alpha^2}\right)\left(\alpha^2-\nu_{0m}{}^2\right)} d\alpha \\ &+ \int_k^\infty \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\sqrt{\nu_{0n}{}^2-k^2}\sqrt{\alpha^2-k^2} - k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\sqrt{\nu_{0n}{}^2-k^2} + \sqrt{\alpha^2-k^2}\right)\left(\alpha^2-\nu_{0m}{}^2\right)} d\alpha = I_1 + I_2 \,. \end{aligned} \tag{7.3}$$

Using (5.5) in the form

$$\frac{i\sqrt{\nu_{0n}{}^2-k^2}\sqrt{k^2-\alpha^2} - k^2}{\sqrt{k^2-\alpha^2}\left(\sqrt{\nu_{0n}{}^2-k^2} + i\sqrt{k^2-\alpha^2}\right)} = \frac{\sqrt{\nu_{0n}{}^2-k^2}\,\alpha^2}{\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)} + i\frac{\nu_{0n}{}^2}{\left(\nu_{0n}{}^2-\alpha^2\right)} \tag{7.4}$$

we write

$$\begin{aligned} I_1 &= \int_0^k \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})}{i\left(\alpha^2-\nu_{0m}{}^2\right)}\left[\frac{\sqrt{\nu_{0n}{}^2-k^2}\,\alpha^2}{\sqrt{k^2-\alpha^2}\left(\nu_{0n}{}^2-\alpha^2\right)} + i\frac{\nu_{0n}{}^2}{\left(\nu_{0n}{}^2-\alpha^2\right)}\right]\alpha d\alpha \\ &= -\nu_{0n}{}^2\int_0^k \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})}{\left(\nu_{0n}{}^2-\alpha^2\right)\left(\nu_{0m}{}^2-\alpha^2\right)}\alpha d\alpha \end{aligned}$$

$$+i\sqrt{{\nu_{0n}}^2-k^2}\int_0^k \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})}{\sqrt{k^2-\alpha^2}\left({\nu_{0n}}^2-\alpha^2\right)\left({\nu_{0m}}^2-\alpha^2\right)}\alpha^3 d\alpha = -{\nu_{0n}}^2 I_{11} + i\sqrt{{\nu_{0n}}^2-k^2}\,I_{12}\,. \qquad (7.5)$$

Using (5.7), we re-write this as

$$I_1 = -{\nu_{0n}}^2\int_0^k \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})}{\left({\nu_{0n}}^2-\alpha^2\right)\left({\nu_{0m}}^2-\alpha^2\right)}\alpha d\alpha$$

$$+i\sqrt{{\nu_{0n}}^2-k^2}\int_0^k \frac{g(\sqrt{k^2-t^2};a,b;\nu_{0n})g(\sqrt{k^2-t^2};a,b;\nu_{0m})\left(k^2-t^2\right)}{\left({\nu_{0n}}^2-k^2+t^2\right)\left({\nu_{0m}}^2-k^2+t^2\right)}dt$$

$$= -{\nu_{0n}}^2 I_{11} + i\sqrt{{\nu_{0n}}^2-k^2}\,I_{12}\,. \qquad (7.6)$$

We turn to the second integral in (7.3)

$$I_2 = \int_k^\infty \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\sqrt{{\nu_{0n}}^2-k^2}+\sqrt{\alpha^2-k^2}\right)\left(\alpha^2-{\nu_{0m}}^2\right)}d\alpha$$

$$= \int_k^{K_m} \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\sqrt{{\nu_{0n}}^2-k^2}+\sqrt{\alpha^2-k^2}\right)\left(\alpha^2-{\nu_{0m}}^2\right)}d\alpha$$

$$+\int_{K_m}^\infty \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\sqrt{{\nu_{0n}}^2-k^2}+\sqrt{\alpha^2-k^2}\right)\left(\alpha^2-{\nu_{0m}}^2\right)}d\alpha = I_{21}+I_{22}\,. \qquad (7.7)$$

As with (5.12), for the first of these integrals, we write

$$I_{21} = \int_k^{K_m} \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\sqrt{\alpha^2-k^2}-k^2\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\sqrt{{\nu_{0n}}^2-k^2}+\sqrt{\alpha^2-k^2}\right)\left(\alpha^2-{\nu_{0m}}^2\right)}d\alpha$$

$$= \int_0^{\sqrt{{K_m}^2-k^2}} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\,t-k^2\right)}{\left(\sqrt{{\nu_{0n}}^2-k^2}+t\right)\left[t^2-\left({\nu_{0m}}^2-k^2\right)\right]}dt$$

$$= \int_0^{\sqrt{{\nu_{0m}}^2-k^2}-10^{-4}} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\,t-k^2\right)}{\left(\sqrt{{\nu_{0n}}^2-k^2}+t\right)\left[t^2-\left({\nu_{0m}}^2-k^2\right)\right]}dt$$

$$+\int_{\sqrt{\nu_{0m}{}^2-k^2}+10^{-4}}^{\sqrt{K_m{}^2-k^2}} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{\nu_{0n}{}^2-k^2}\,t-k^2\right)}{\left(\sqrt{\nu_{0n}{}^2-k^2}+t\right)\left[t^2-\left(\nu_{0m}{}^2-k^2\right)\right]}dt \tag{7.8}$$

while for the second

$$I_{22}=\int_{K_m}^{\infty} \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)\left(\alpha^2-\nu_{0m}{}^2\right)}d\alpha\,. \tag{7.9}$$

As with Sections 5 and 6, we make the upper limit of integration in the second integral in (7.8) be an integer. We can then write

$$I_2=\int_0^{\sqrt{\nu_{0m}{}^2-k^2}-10^{-4}} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{\nu_{0n}{}^2-k^2}\,t-k^2\right)}{\left(\sqrt{\nu_{0n}{}^2-k^2}+t\right)\left[t^2-\left(\nu_{0m}{}^2-k^2\right)\right]}dt$$

$$+\int_{\sqrt{\nu_{0m}{}^2-k^2}+10^{-4}}^{L} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{\nu_{0n}{}^2-k^2}\,t-k^2\right)}{\left(\sqrt{\nu_{0n}{}^2-k^2}+t\right)\left[t^2-\left(\nu_{0m}{}^2-k^2\right)\right]}dt$$

$$+\int_{\sqrt{L^2+k^2}}^{\infty} \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)\left(\alpha^2-\nu_{0m}{}^2\right)}d\alpha\,. \tag{7.10}$$

The last integral is computed asymptotically. We can approximate the integrand by

$$\frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)\left(\alpha^2-\nu_{0m}{}^2\right)}$$

$$=\frac{\sqrt{\nu_{0n}{}^2-k^2}}{\pi\alpha^3}\left[\frac{1}{aY_0(\nu_{0m}a)Y_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0m}b)Y_0(\nu_{0n}b)}\right]+O(\alpha^{-4})\,. \tag{7.11}$$

Then

$$\int_{\sqrt{L^2+k^2}}^{\infty} \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})\left(\alpha^2\sqrt{\nu_{0n}{}^2-k^2}-\nu_{0n}{}^2\sqrt{\alpha^2-k^2}\right)\alpha}{\sqrt{\alpha^2-k^2}\left(\alpha^2-\nu_{0n}{}^2\right)\left(\alpha^2-\nu_{0m}{}^2\right)}d\alpha$$

$$\approx\frac{\sqrt{\nu_{0n}{}^2-k^2}}{2\pi\left(L^2+k^2\right)}\left[\frac{1}{aY_0(\nu_{0m}a)Y_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0m}b)Y_0(\nu_{0n}b)}\right]. \tag{7.12}$$

From (7.6), (7.8), (7.9), and (7.12) we have for the fourth, and last, integral

$$
\begin{aligned}
I \approx & -{\nu_{0n}}^2 \int_0^k \frac{g(\alpha;a,b;\nu_{0n})g(\alpha;a,b;\nu_{0m})}{\left({\nu_{0n}}^2-\alpha^2\right)\left({\nu_{0m}}^2-\alpha^2\right)}\alpha d\alpha \\
& + i\sqrt{{\nu_{0n}}^2-k^2}\int_0^k \frac{g(\sqrt{k^2-t^2};a,b;\nu_{0n})g(\sqrt{k^2-t^2};a,b;\nu_{0m})\left(k^2-t^2\right)}{\left({\nu_{0n}}^2-k^2+t^2\right)\left({\nu_{0m}}^2-k^2+t^2\right)}dt \\
& + \int_0^{\sqrt{{\nu_{0m}}^2-k^2}-10^{-4}} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\,t-k^2\right)}{\left(\sqrt{{\nu_{0n}}^2-k^2}+t\right)\left[t^2-\left({\nu_{0m}}^2-k^2\right)\right]}dt \\
& + \int_{\sqrt{{\nu_{0m}}^2-k^2}+10^{-4}}^{L} \frac{g(\sqrt{t^2+k^2};a,b;\nu_{0n})g(\sqrt{t^2+k^2};a,b;\nu_{0m})\left(\sqrt{{\nu_{0n}}^2-k^2}\,t-k^2\right)}{\left(\sqrt{{\nu_{0n}}^2-k^2}+t\right)\left[t^2-\left({\nu_{0m}}^2-k^2\right)\right]}dt \\
& + \frac{\sqrt{{\nu_{0n}}^2-k^2}}{2\pi\left(L^2+k^2\right)}\left[\frac{1}{aY_0(\nu_{0m}a)Y_0(\nu_{0n}a)}+\frac{1}{bY_0(\nu_{0m}b)Y_0(\nu_{0n}b)}\right]. \qquad (7.13)
\end{aligned}
$$

APPENDIX: EVALUATION OF INTEGRALS FOR SECTION 3

We first observe that, from (2.5) and (2.7)

$$\int_a^b P_{0m}{}'(\rho)d\rho = P_{0m}(\rho)\Big|_a^b = 0\,. \tag{A.1}$$

We next evaluate the integral

$$\begin{aligned}\int_a^b \frac{\partial P_{0m}(\rho)}{\partial\rho}\frac{\partial J_0(\alpha\rho)}{\partial\rho}\rho d\rho &= \int_a^b \frac{\partial}{\partial\rho}\left[J_0(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_0(\nu_{0m}\rho)\right]\frac{\partial J_0(\alpha\rho)}{\partial\rho}\rho d\rho \\ &= \alpha\nu_{0m}\int_a^b\left[J_1(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}\rho)\right]J_1(\alpha\rho)\rho d\rho \\ &= \alpha\nu_{0m}\left\{\int_a^b J_1(\nu_{0m}\rho)J_1(\alpha\rho)\rho d\rho - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}\int_a^b Y_1(\nu_{0m}\rho)J_1(\alpha\rho)\rho d\rho\right\}.\end{aligned} \tag{A.2}$$

Using standard formulas (reference 3, p. 87), we proceed to evaluate the last two integrals

$$\begin{aligned}&\int_a^b \frac{\partial P_{0m}(\rho)}{\partial\rho}\frac{\partial J_0(\alpha\rho)}{\partial\rho}\rho d\rho = \frac{\alpha\nu_{0m}}{\nu_{0m}{}^2-\alpha^2}\rho\Big\{\nu_{0m}J_2(\nu_{0m}\rho)J_1(\alpha\rho) - \alpha J_1(\nu_{0m}\rho)J_2(\alpha\rho) \\ &-\frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}\left[\nu_{0m}Y_2(\nu_{0m}\rho)J_1(\alpha\rho) - \alpha Y_1(\nu_{0m}\rho)J_2(\alpha\rho)\right]\Big\}\Big|_a^b \\ &= \frac{\alpha\nu_{0m}}{\nu_{0m}{}^2-\alpha^2}\rho\Big\{\nu_{0m}\left[J_2(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_2(\nu_{0m}\rho)\right]J_1(\alpha\rho) \\ &-\alpha\left[J_1(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}\rho)\right]J_2(\alpha\rho)\Big\}\Big|_a^b \\ &= \frac{\alpha\nu_{0m}\rho}{\nu_{0m}{}^2-\alpha^2}\left\{\frac{2}{\rho}\left[J_1(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}\rho)\right] - \nu_{0m}\left[J_0(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_0(\nu_{0m}\rho)\right]\right\}J_1(\alpha\rho)\Big|_a^b \\ &-\frac{\alpha^2\nu_{0m}}{\nu_{0m}{}^2-\alpha^2}\rho\left[J_1(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}\rho)\right]\left[\frac{2}{\alpha\rho}J_1(\alpha\rho) - J_0(\alpha\rho)\right]\Big|_a^b \\ &= \frac{\alpha^2\nu_{0m}}{\nu_{0m}{}^2-\alpha^2}\rho\left[J_1(\nu_{0m}\rho) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}\rho)\right]J_0(\alpha\rho)\Big|_a^b \\ &= \frac{\alpha^2\nu_{0m}}{\nu_{0m}{}^2-\alpha^2}\left\{b\left[J_1(\nu_{0m}b) - \frac{J_0(\nu_{0m}b)}{Y_0(\nu_{0m}b)}Y_1(\nu_{0m}b)\right]J_0(\alpha b) - a\left[J_1(\nu_{0m}a) - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}a)\right]J_0(\alpha a)\right\}\end{aligned} \tag{A.3}$$

or

$$\int_a^b \frac{\partial P_{0m}(\rho)}{\partial\rho}\frac{\partial J_0(\alpha\rho)}{\partial\rho}\rho d\rho = \frac{2\alpha^2}{\pi\left(\alpha^2-\nu_{0m}{}^2\right)}\left[\frac{J_0(\alpha a)}{Y_0(\nu_{0m}a)}-\frac{J_0(\alpha b)}{Y_0(\nu_{0m}b)}\right] \tag{A.4}$$

Finally, we compute

$$\begin{aligned}
&\int_a^b \frac{\partial P_{0m}(\rho)}{\partial\rho}\frac{\partial P_{0n}(\alpha\rho)}{\partial\rho}\rho d\rho \\
&= \nu_{0m}\nu_{0n}\int_a^b\left[J_1(\nu_{0m}\rho)-\frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}Y_1(\nu_{0m}\rho)\right]\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]\rho d\rho \\
&= \nu_{0m}\nu_{0n}\left\{\int_a^b J_1(\nu_{0m}\rho)J_1(\nu_{0n}\rho)\rho d\rho - \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}\int_a^b Y_1(\nu_{0m}\rho)J_1(\nu_{0n}\rho)\rho d\rho\right. \\
&\left. -\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\int_a^b J_1(\nu_{0m}\rho)Y_1(\nu_{0n}\rho)\rho d\rho + \frac{J_0(\nu_{0m}a)}{Y_0(\nu_{0m}a)}\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\int_a^b Y_1(\nu_{0m}\rho)Y_1(\nu_{0n}\rho)\rho d\rho\right\}.
\end{aligned} \tag{A.5}$$

If we proceed to complete this calculation, we will find that the result is zero. On the other hand, if $m = n$, then

$$\begin{aligned}
&\int_a^b\left[\frac{\partial P_{0n}(\alpha\rho)}{\partial\rho}\right]^2\rho d\rho = \nu_{0n}{}^2\int_a^b\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]^2\rho d\rho \\
&= \nu_{0n}{}^2\left\{\int_a^b\left[J_1(\nu_{0n}\rho)\right]^2\rho d\rho - 2\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\int_a^b J_1(\nu_{0n}\rho)Y_1(\nu_{0n}\rho)\rho d\rho\right. \\
&\qquad\left. +\left[\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\right]^2\int_a^b\left[Y_1(\nu_{0n}\rho)\right]^2\rho d\rho\right\}.
\end{aligned} \tag{A.6}$$

Proceeding with the same kind of formulas as above, we find

$$\begin{aligned}
&\int_a^b\left[\frac{\partial P_{0n}(\alpha\rho)}{\partial\rho}\right]^2\rho d\rho \\
&= \nu_{0n}{}^2\left\{\frac{\rho^2}{2}\left[\left(J_1(\nu_{0n}\rho)\right)^2 - J_2(\nu_{0n}\rho)J_0(\nu_{0n}\rho) + \left[\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\right]^2\left\{\left(Y_1(\nu_{0n}\rho)\right)^2 - Y_2(\nu_{0n}\rho)Y_0(\nu_{0n}\rho)\right\}\right]\right. \\
&\qquad\left.-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\frac{\rho^2}{2}\left[2J_1(\nu_{0n}\rho)Y_1(\nu_{0n}\rho) - J_2(\nu_{0n}\rho)Y_0(\nu_{0n}\rho) - J_0(\nu_{0n}\rho)Y_2(\nu_{0n}\rho)\right]\right\}\Bigg|_a^b \\
&= \frac{\nu_{0n}{}^2\rho^2}{2}\left\{\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]J_1(\nu_{0n}\rho) - \left[J_0(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_0(\nu_{0n}\rho)\right]J_2(\nu_{0n}\rho)\right.
\end{aligned}$$

$$-\left[\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\right]\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]Y_1(\nu_{0n}\rho)$$

$$+\left[\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\right]\left[J_0(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_0(\nu_{0n}\rho)\right]Y_2(\nu_{0n}\rho)\Bigg\}\Bigg|_a^b$$

$$=\frac{\nu_{0n}{}^2\rho^2}{2}\left\{\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]J_1(\nu_{0n}\rho)\right.$$

$$\left.-\left[\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}\right]\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]Y_1(\nu_{0n}\rho)\right\}\Bigg|_a^b$$

$$=\frac{\nu_{0n}{}^2\rho^2}{2}\left[J_1(\nu_{0n}\rho)-\frac{J_0(\nu_{0n}a)}{Y_0(\nu_{0n}a)}Y_1(\nu_{0n}\rho)\right]^2\Bigg|_a^b$$

$$=\frac{\nu_{0n}{}^2}{2}\left\{\left[\frac{b}{Y_0(\nu_{0n}b)}\frac{2}{\pi\nu_{0n}b}\right]^2-\left[\frac{a}{Y_0(\nu_{0n}a)}\frac{2}{\pi\nu_{0n}a}\right]^2\right\} \tag{A.7}$$

or

$$\int_a^b\left[\frac{\partial P_{0n}(\alpha\rho)}{\partial\rho}\right]^2\rho d\rho=\frac{2}{\pi^2}\left\{\frac{1}{\left[Y_0(\nu_{0n}b)\right]^2}-\frac{1}{\left[Y_0(\nu_{0n}a)\right]^2}\right\}. \tag{A.8}$$

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PART 4
NOTES ON COMPUTING THE INFINITE SYSTEM OF EQUATIONS

ABSTRACT

This is the fourth, and final, part of the report on the formulation of the problem of radiation of a coaxial line into a half space in terms of BIEs. In it, we give a detailed presentation of how we compute the infinite system of equations. Specifically, we truncate the infinite system and consider finite systems of dimensions from 1x1 to 11x11. We examine the convergence of the coefficients of these systems for four different coaxial lines and use them in computing far-field quantities, such as directivity and gain.

1. INTRODUCTION: WHAT IS TO BE COMPUTED

We use here the results of Part 3 to compute the radiated fields of a coaxial transmission line that opens into a half space. We use Mathematica® (reference 1) in all computations. In this section we assign values to all relevant parameters.

We set the characteristic impedance of the coaxial line at 50 Ω; thus,

$$50 = Z_c = Z_0 \frac{\ln(\chi)}{2\pi} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln(b/a)}{2\pi} . \tag{1.1}$$

From this

$$\chi = \frac{b}{a} = \mathrm{e}^{100\pi\sqrt{\varepsilon_0/\mu_0}} = 2.3023 \tag{1.2}$$

or

$$\rho = \frac{a}{b} = 0.4343 . \tag{1.3}$$

For all modes (except the TEM) to be suppressed

$$k\frac{a+b}{2} < 1 \tag{1.4}$$

or

$$kb < 1.3944 \quad \text{or} \quad \frac{b}{\lambda} < 0.2219 . \tag{1.5}$$

We set $\lambda = 1$ (300 MHz) and give b the values shown in Table 1.1.

Table 1.1. Values of the parameters for the numerical computation ($\lambda = 1$).

Case	b (m)	a (m)	kb	ka
1	0.05	0.021715	0.314159	0.136439
2	0.10	0.043430	0.628319	0.272879
3	0.15	0.065145	0.942478	0.409318
4	0.20	0.086860	1.256637	0.545757

2. FIRST INTEGRAL

In this section, we compute the integral in (4.22) of Part 3. The imaginary part is the sum of two integrals. Both were computed in Mathematica® (reference 1). Their sum is given in Table 2.1 for the four different coaxial lines in Table 1.1.

Table 2.1. Numerical results for integral I_{11}, the imaginary part of (4.22).

Case	I_{11}
1	- 0.0000420305 i
2	- 0.000649198 i
3	- 0.00309862 i
4	- 0.00901561 i

The result for the second line in (4.22) is shown in Figure 2.2.

Table 2.2. Numerical results for integral I_{12}, the real part of (4.22).

Case	I_{12}
1	0.0136816
2	0.0285517
3	0.0452372
4	0.0634187

The entire integral in (4.22) is shown in Table 2.3.

Table 2.3. Numerical results for integral I_1, the entire expression in (4.22).

Case	I_1
1	0.0136816 - 0.0000420305 i
2	0.0285517 - 0.000649198 i
3	0.0452372 - 0.00309862 i
4	0.0634187 - 0.00901561 i

3. SECOND INTEGRAL

The second integral appears in (5.20) of Part 3. We compute this integral for the four cases in Table 1.1 and the first 10 roots of the transcendental equation

$$J_0(\nu_{0n}a)Y_0(\nu_{0n}b) - J_0(\nu_{0n}b)Y_0(\nu_{0n}a) = 0\,. \tag{3.1}$$

The roots for the four cases have been computed in Mathematica® (reference 1). We observe that each successive case (column) is equal to the first one divided by the number of the column. We note that Mathematica® has a subroutine that finds the roots of this equation but we did not take advantage of it.

Table 3.1. First ten roots of the transcendental equation (3.1)

Root No.	Case 1	Case 2	Case 3	Case 4
1	110.131	55.0656	36.7104	27.5328
2	221.636	110.818	73.8786	55.4089
3	332.867	166.434	110.956	83.2168
4	444.02	222.01	148.007	111.005
5	555.14	277.57	185.047	138.785
6	666.243	333.122	222.081	166.561
7	777.337	388.668	259.112	194.334
8	888.424	444.212	296.141	222.106
9	999.508	499.754	333.169	249.877
10	1110.59	555.294	370.196	277.647

The entire expression (5.20) is computed in Mathematica®. The results are displayed in Tables 3.2.1 through 3.2.5.

Table 3.2.1 Numerical results for integral I_2, the expression (5.20) (roots 1 and 2)

Case	Root	
	1	2
1	1.26328 + 0.0036659 i	- 2.07424 + 0.000668069 i
2	1.22771 + 0.0277552 i	- 2.06343 + 0.00443626 i
3	1.18825 + 0.0853679 i	- 2.03808 + 0.0109216 i
4	1.16772 + 0.177285 i	- 1.99193 + 0.0109216 i

Table 3.2.2 Numerical results for integral I_2, the expression (5.20) (roots 3 and 4)

Case	Root	
	3	4
1	17.9066 + 0.0147716 i	4.7176 – 0.000653382 i
2	17.7639 + 0.111543 i	4.70708 – 0.00433879 i
3	17.6066 + 0.341594 i	4.68246 – 0.0100815 i
4	17.5259 + 0.705258 i	4.63767 – 0.0107471 i

Table 3.2.3 Numerical results for integral I_2, the expression (5.20) (roots 5 and 6)

Case	Root	
	5	6
1	- 3.52255 - 0.00168172 i	3.76781 - 0.00032758 i
2	- 3.50631 - 0.0126963 i	3.76253 - 0.00217531 i
3	- 3.48842 - 0.0388686 i	3.7502 - 0.00505523 i
4	- 3.47925 - 0.0802122 i	3.72778 - 0.00539464 i

Table 3.2.4 Numerical results for integral I_2, the expression (5.20) (roots 7 and 8)

Case	Root	
	7	8
1	- 11.0512 - 0.00370696 i	- 11.7554 + 0.00074151 i
2	- 11.0154 - 0.0279843 i	- 11.7434 + 0.00492407 i
3	- 10.9759 - 0.0856635 i	- 11.7155 + 0.0114437 i
4	- 10.9557 - 0.17676 i	- 11.6648 + 0.0122167 i

Table 3.2.5 Numerical results for integral I_2, the expression (5.20) (roots 9 and 10)

Case	Root	
	9	10
1	5.31419 + 0.00137324 i	- 4.72606 + 0.000233462 i
2	5.30092 + 0.0103665 i	- 4.72226 + 0.00155033 i
3	5.28632 + 0.031732 i	- 4.71348 + 0.00360309 i
4	5.27883 + 0.0654731 i	- 4.69751 + 0.0038471 i

4. THE THIRD INTEGRAL

The third integral is given by (6.3) in Part 3. The entire expression is computed in Mathematica®. The results are displayed in Tables 4.1.1 through 4.1.5.

Table 4.1.1 Numerical results for integral I_3, the expression (6.3) (roots 1 and 2)

Case	Root	
	1	2
1	- 0.00354536 + 0.0000333409 i	0.00433068 + 3.01548*10^{-6} i
2	- 0.00772607 + 0.000507352 i	0.00875998 + 0.0000400965 i
3	- 0.0141551 + 0.00236027 i	0.0136065 + 0.000139937 i
4	- 0.0219913 + 0.00661357 i	0.0191409 + 0.000198389 i

Table 4.1.2 Numerical results for integral I_3, the expression (6.3) (roots 3 and 4)

Case	Root	
	3	4
1	- 0.0057484 + 0.0000443849 i	- 0.00269068 - 1.47166*10^{-6} i
2	- 0.0119776 + 0.000670672 i	- 0.00543566 - 0.0000195511 i
3	- 0.0199091 + 0.0030836 i	- 0.00797123 - 0.0000681768 i
4	- 0.026884 + 0.00849921 i	- 0.011534 - 0.0000969721 i

Table 4.1.3 Numerical results for integral I_3, the expression (6.3) (roots 5 and 6)

Case	Root	
	5	6
1	0.000420177 - 3.02957*10^{-6} i	- 0.00102198 - 4.91704*10^{-7} i
2	0.000916501 - 0.0000457526 i	- 0.00206025 - 6.53125*10^{-6} i
3	0.0014815 - 0.000210169 i	- 0.00314133 - 0.0000227721 i
4	0.00234308 - 0.000578554 i	- 0.00434858 - 0.0000324115 i

Table 4.1.4 Numerical results for integral I_3, the expression (6.3) (roots 7 and 8)

Case	Root	
	7	8
1	0.000739879 - 4.76895*10^{-6} i	0.00185938 + 8.34656*10^{-7} i
2	0.00154382 - 0.0000720099 i	0.00374666 + 0.000011086 i
3	0.00231508 - 0.000330701 i	0.00562887 + 0.0000386513 i
4	0.00340646 - 0.000910044 i	0.00785099 + 0.0000550258 i

Table 4.1.5 Numerical results for integral I_3, the expression (6.3) (roots 9 and 10)

Case	Root	
	9	10
1	- 0.000201699 + 1.37394*10^{-6} i	0.000498893 + 2.10218*10^{-7} i
2	- 0.000431858 + 0.0000207448 i	0.00100607 + 2.79208*10^{-6} i
3	- 0.000688448 + 0.0000952597 i	0.0015313 + 9.73432*10^{-6} i
4	- 0.00108081 + 0.000262104 i	0.00210175 + 0.0000138596 i

5. THE FOURTH INTEGRAL

The fourth integral is given by (7.1) in Part 3, and in computable form by (7.13). The result in (7.13) depends not only on the index *m* but also on the index *n*. Since *n* runs from 1 to 10, we have 10 times as much data as in the previous cases. For this reason, we do not form tables but, rather we display the raw data from Mathematica®. For the first 10 roots, we have

```
{{{-0.566603+0.002908 i,0.00857833 +0.000529959 i,-
5.38481+0.0117177 i,-0.0107252-0.000518308 i,0.81978 -0.00133404
i,-0.00604899-0.000259859 i,2.12689 -0.00294057 i,0.0147767
+0.000588217 i,-0.879104+0.00108933 i,0.00491084 +0.000185199
i},{0.00425818 +0.000263014 i,-0.840763+0.0000479381 i,0.0279246
+0.00105982 i,1.54291 -0.0000468842 i,-0.00367476-0.000120658
i,1.03996 -0.0000235059 i,-0.00880162-0.000265962 i,-
2.82511+0.000053208 i,0.00344896 +0.0000985254 i,-
1.01075+0.0000167524 i},{-1.77903+0.00387126 i,0.0185789
+0.00070551 i,-25.442+0.0155992 i,-0.0222245-0.00069 i,4.5937 -
0.00177594 i,-0.0122724-0.000345938 i,13.1365 -0.00391463
i,0.0295536 +0.000783066 i,-5.7899+0.00145017 i,0.00974883
+0.000246546 i},{-0.00265561-0.000128361 i,0.769935 -0.0000233956
i,-0.0166423-0.00051723 i,-1.64689+0.0000228813 i,0.00215396
+0.0000588857 i,-1.20977+0.0000114717 i,0.00510935 +0.000129799
i,3.47599 -0.0000259675 i,-0.00198939-0.000048084 i,1.29426 -
8.17578×10^-6 i},{0.162381 -0.000264239 i,-0.00146457-0.0000481557
i,2.75418 -0.00106475 i,0.00172414 +0.0000470971 i,-
0.543394+0.000121219 i,0.000942938 +0.0000236126 i,-
1.64399+0.000267199 i,-0.00226252-0.0000534495 i,0.753789 -
0.0000989835 i,-0.000741743-0.0000168284 i},{-0.000998465-
0.0000428871 i,0.345853 -7.81679×10^-6 i,-0.00613222-0.000172814
i,-0.806248+7.64495×10^-6 i,0.000787625 +0.0000196745 i,-
0.625213+3.83287×10^-6 i,0.00186647 +0.0000433678 i,1.86557 -
8.6761×10^-6 i,-0.000725125-0.0000160655 i,0.71443 -2.73165×10^-6
i},{0.300869 -0.000415949 i,-0.00250545-0.0000758037 i,5.62488 -
0.00167606 i,0.00292745 +0.0000741373 i,-1.1741+0.000190816
i,0.00159898 +0.0000371694 i,-3.69107+0.000420608 i,-0.00382792-
0.0000841368 i,1.74081 -0.000155814 i,-0.00125978-0.0000264902
i},{0.00182884 +0.0000727999 i,-0.704602+0.0000132688 i,0.0110647
+0.000293348 i,1.73733 -0.0000129771 i,-0.00141593-0.0000333971
i,1.39912 -6.50621×10^-6 i,-0.00334119-0.0000736159 i,-
4.29179+0.0000147275 i,0.00129871 +0.0000272709 i,-
1.67888+4.63691×10^-6 i},{-0.0967212+0.000119835 i,0.000762736
+0.0000218391 i,-1.92827+0.000482874 i,-0.000885603-0.000021359
i,0.418727 -0.0000549743 i,-0.000481306-0.0000107086 i,1.35405 -
0.000121178 i,0.00115215 +0.0000242399 i,-0.652457+0.0000448901
i,0.000377498 +7.63187×10^-6 i},{0.000486278 +0.0000183356 i,-
0.201679+3.34192×10^-6 i,0.00292129 +0.0000738832 i,0.51754 -
3.26845×10^-6 i,-0.000372048-8.41147×10^-6 i,0.428674 -1.63867×10^-6
i,-0.000880994-0.0000185411 i,-1.34322+3.7093×10^-6 i,0.000341725
+6.86851×10^-6 i,-0.53434+1.16786×10^-6 i}},{{-0.544135+0.0216915
i,0.00803394 +0.00346806 i,-5.29309+0.0871752 i,-0.0102082-
0.00339188 i,0.809014 -0.00992268 i,-0.00579225-0.00170056
i,2.10209 -0.0218709 i,0.0141846 +0.00384942 i,-
```

0.869407+0.00810185 i,0.00472671 +0.00121198 i},{0.00397402 +0.00171479 i,-0.835292+0.000274929 i,0.0268598 +0.00689244 i,1.53667 -0.000268894 i,-0.00355686-0.000784538 i,1.0361 -0.000134814 i,-0.00853931-0.00172923 i,-2.81408+0.000305168 i,0.00335349 +0.000640577 i,-1.00638+0.0000960814 i},{-1.74106+0.0286746 i,0.0178479 +0.00458514 i,-25.2715+0.11524 i,-0.0215319-0.00448442 i,4.57028 -0.0131171 i,-0.011922-0.00224833 i,13.072 -0.0289119 i,0.0287416 +0.00508934 i,-5.75974+0.0107101 i,0.00948526 +0.00160237 i},{-0.0025154-0.000836135 i,0.766117 -0.000134058 i,-0.0161017-0.00336077 i,-1.64139+0.000131116 i,0.00209031 +0.000382542 i,-1.20557+0.0000657369 i,0.00495753 +0.000843177 i,3.46201 -0.000148803 i,-0.00192969-0.000312347 i,1.28809 -0.0000468503 i},{0.159489 -0.00195615 i,-0.001414-0.000312798 i,2.73914 -0.00786156 i,0.00167437 +0.000305926 i,-0.540922+0.000894839 i,0.000917494 +0.00015338 i,-1.63608+0.00197235 i,-0.00219693-0.000347194 i,0.749676 -0.000730635 i,-0.000720989-0.000109313 i},{-0.000951622-0.00027932 i,0.344182 -0.0000447838 i,-0.00595562-0.0011227 i,-0.803284+0.0000438008 i,0.000768246 +0.000127793 i,-0.622639+0.0000219602 i,0.001819 +0.000281673 i,1.85637 -0.0000497095 i,-0.00070751-0.000104343 i,0.710204 -0.0000156509 i},{0.295914 -0.00307878 i,-0.00242454-0.000492313 i,5.5944 -0.0123733 i,0.0028485 +0.000481498 i,-1.16828+0.00140839 i,0.00155717 +0.000241406 i,-3.67066+0.00310427 i,-0.0037231-0.000546449 i,1.72959 -0.00114995 i,-0.0012235-0.000172048 i},{0.00174669 +0.000474114 i,-0.70098+0.0000760155 i,0.0107399 +0.00190566 i,1.72976 -0.000074347 i,-0.0013748-0.000216913 i,1.39203 -0.000037275 i,-0.00323485-0.000478107 i,-4.26555+0.0000843763 i,0.0012527 +0.00017711 i,-1.66659+0.0000265657 i},{-0.0951797+0.000886947 i,0.000738688 +0.000141827 i,-1.91701+0.00356454 i,-0.000859986-0.000138712 i,0.416318 -0.000405732 i,-0.000466894-0.0000695451 i,1.3451 -0.000894289 i,0.00111042 +0.000157423 i,-0.64739+0.00033128 i,0.00036287 +0.0000495643 i},{0.000465729 +0.000119408 i,-0.200534+0.000019145 i,0.0028385 +0.000479952 i,0.514829 -0.0000187247 i,-0.000362402-0.0000546309 i,0.426016 -9.38793×10^{-6} i,-0.000853457-0.000120414 i,-1.33318+0.0000212507 i,0.000330501 +0.0000446062 i,-0.529591+6.69072×10^{-6} i}},{{-0.523709+0.0650368 i,0.0022977 +0.00786479 i,-5.21129+0.26026 i,-0.00464348-0.00769733 i,0.799647 -0.0296142 i,-0.00300696-0.00385971 i,2.08124 -0.0652676 i,0.00788145 +0.00873736 i,-0.861588+0.0241768 i,0.00274315 +0.00275099 i},{0.00113137 +0.00386437 i,-0.826442+0.000480635 i,0.0156485 +0.0154798 i,1.5279 -0.000470437 i,-0.00228503-0.00176153 i,1.03156 -0.000235897 i,-0.00574097-0.00388238 i,-2.80337+0.000534009 i,0.002319 +0.00143815 i,-1.00283+0.000168135 i},{-1.70148+0.0849746 i,0.0103677 +0.0102862 i,-25.1102+0.340059 i,-0.0142872-0.0100672 i,4.55118 -0.0386942 i,-0.00829791-0.00504806 i,13.0274 -0.0852795 i,0.0205462 +0.0114275 i,-5.74204+0.0315897 i,0.00690648 +0.00359798 i},{-0.00113422-0.00188271 i,0.760584 -0.000234183 i,-0.0106512-0.00754175 i,-1.63568+0.000229214 i,0.00146939 +0.000858217 i,-1.20244+0.000114937 i,0.00358463 +0.00189149 i,3.45405 -0.000260189 i,-0.00141951-0.000700664 i,1.28524 -0.0000819216 i},{0.156387 -0.00579165 i,-0.000903337-

0.000701136 i,2.72612 -0.0231776 i,0.00117898 +0.000686209 i,-
0.539298+0.0026373 i,0.000669242 +0.000344089 i,-
1.63204+0.00581244 i,-0.00163336-0.000778926 i,0.74795 -
0.00215308 i,-0.000543438-0.000245248 i},{-0.000490266-
0.000628855 i,0.342058 -0.0000782216 i,-0.00414459-0.00251906 i,-
0.800967+0.000076562 i,0.000563258 +0.000286658 i,-
0.621272+0.0000383913 i,0.00136826 +0.000631788 i,1.85263 -
0.0000869081 i,-0.000541296-0.000234033 i,0.708767 -0.0000273634
i},{0.290601 -0.00911319 i,-0.00162162-0.00110327 i,5.57123 -
0.0364701 i,0.00207271 +0.00107978 i,-1.16521+0.00414982
i,0.00116998 +0.000541438 i,-3.66245+0.00914592 i,-0.00284932-
0.00122567 i,1.72585 -0.00338789 i,-0.000949291-0.000385908
i},{0.000962164 +0.00106736 i,-0.696984+0.000132767 i,0.00765227
+0.00427563 i,1.72511 -0.00012995 i,-0.00102291-0.000486548
i,1.38908 -0.0000651623 i,-0.002456-0.00107234 i,-
4.25691+0.000147511 i,0.0009628 +0.000397226 i,-
1.66309+0.0000464444 i},{-0.0935508+0.00262509 i,0.000506765
+0.000317803 i,-1.90956+0.0105053 i,-0.000634578-0.000311037
i,0.415259 -0.00119537 i,-0.000353581-0.000155965 i,1.34207 -
0.00263452 i,0.000851885 +0.000353063 i,-0.645931+0.000975894
i,0.000281057 +0.000111163 i},{0.000268097 +0.000268815 i,-
0.199436+0.0000334374 i,0.00206333 +0.00107682 i,0.513462 -
0.0000327279 i,-0.000274645-0.000122537 i,0.425085 -0.0000164111
i,-0.000659985-0.000270069 i,-1.3303+0.0000371506 i,0.000259113
+0.000100041 i,-0.528378+0.000011697 i}},{{-0.523207+0.13013 i,-
0.0114275+0.00814608 i,-5.20929+0.517813 i,0.00867961 -0.00801511
i,0.799399 -0.0588946 i,0.00366306 -0.00402319 i,2.08062 -
0.129784 i,-0.00720973+0.00911082 i,-0.861324+0.048073 i,-
0.00200699+0.00286904 i},{-0.00556127+0.00396658 i,-
0.812806+0.000350881 i,-0.0106277+0.0158966 i,1.5146 -0.000344593
i,0.00069799 -0.00180902 i,1.02485 -0.000172904 i,0.000825982 -
0.00398707 i,-2.78805+0.000391499 i,-0.000111212+0.00147693 i,-
0.997943+0.000123278 i},{-1.68285+0.167279 i,-
0.00709115+0.0105464 i,-25.0354+0.665716 i,0.00264411 -0.0103764
i,4.54241 -0.0757173 i,0.000175133 -0.00520837 i,13.0072 -
0.166856 i,0.00138282 +0.0117947 i,-5.73413+0.0618049
i,0.000876203 +0.00371421 i},{0.00210141 -0.00193866 i,0.752356 -
0.000171171 i,0.00204888 -0.00776907 i,-1.62757+0.000168105
i,0.000025202 +0.00088411 i,-1.19828+0.0000843491 i,0.00039908
+0.00194857 i,3.44434 -0.000190989 i,-0.000238183-0.000721811
i,1.28206 -0.0000601397 i},{0.154562 -0.0113872 i,0.000285326 -
0.000718316 i,2.71868 -0.0453177 i,0.0000255571 +0.000706732 i,-
0.538402+0.00515435 i,0.0000914928 +0.000354741 i,-
1.6299+0.0113585 i,-0.000324936-0.000803336 i,0.747076 -
0.00420728 i,-0.000131303-0.000252974 i},{0.000589496 -
0.000647953 i,0.338976 -0.0000571887 i,0.0000823688 -0.00259662
i,-0.797887+0.0000561646 i,0.0000838364 +0.000295491 i,-
0.619656+0.0000281813 i,0.000313557 +0.000651262 i,1.84875 -
0.0000638099 i,-0.000151385-0.000241247 i,0.707452 -0.0000200929
i},{0.28715 -0.0179117 i,0.000247122 -0.00113006 i,5.55689 -
0.0712836 i,0.000262664 +0.00111184 i,-1.16342+0.00810767
i,0.000265133 +0.000558081 i,-3.65799+0.0178667 i,-0.000805087-
0.00126381 i,1.72394 -0.00661796 i,-0.000306815-0.000397979 i},{-
0.000871399+0.00110004 i,-0.691327+0.0000970764 i,0.00468113

```
+0.00440829 i,1.71936 -0.000095338 i,-0.000206034-0.000501657
i,1.38597 -0.0000478371 i,-0.000653892-0.00110565 i,-
4.2492+0.000108316 i,0.000294315 +0.000409566 i,-
1.66038+0.0000341072 i},{-0.0924306+0.00515881 i,-
0.0000320495+0.000325492 i,-1.9048+0.0205307 i,-0.000111555-
0.000320243 i,0.414644 -0.00233512 i,-0.0000913684-0.000160745
i,1.34047 -0.00514585 i,0.000257298 +0.000364017 i,-
0.64522+0.00190606 i,0.000093544 +0.000114631 i},{-
0.000193758+0.000277071 i,-0.197922+0.0000244496 i,0.000256923
+0.00111033 i,0.511888 -0.0000240118 i,-0.0000697322-0.000126354
i,0.424209 -0.0000120483 i,-0.000208931-0.000278484 i,-
1.32805+0.0000272804 i,0.0000923197 +0.000103159 i,-
0.527559+8.59024×10^-6 i}}}
```

6. THE SYSTEM OF EQUATIONS

The infinite system of equations is given by (3.13) and (3.14) in Part 3. We can solve it by truncating it to a finite system of equations. What we have done is to work with a 1x1 system (returning TEM mode only) up to a 11x11 system. For systems of order higher than this, we need to compute the integrals above for more roots of the transcendental equation (3.1).

Since we stop with a 11x11 system, we examine the convergence of the first four coefficients as a function of the order of the system for each of the four coaxial lines in Table 1.1. In Figures 6.1 and 6.2, we display the real and imaginary part, respectively, of the coefficient *A* as a function of the order of the system. We see that, for engineering purposes, *A* has stabilized for the 11x11 system and for all four coaxials. If we put these results under the microscope, however, we may conclude that the data has converged to at most two significant digits, as seen from Figures 6.3 and 6.4.

We see that, as the radii of the coax get larger, the reflection coefficient becomes smaller in magnitude. This means that more energy escapes into the upper-half space. This will become evident when we compute the gain of the coaxial line as an antenna.

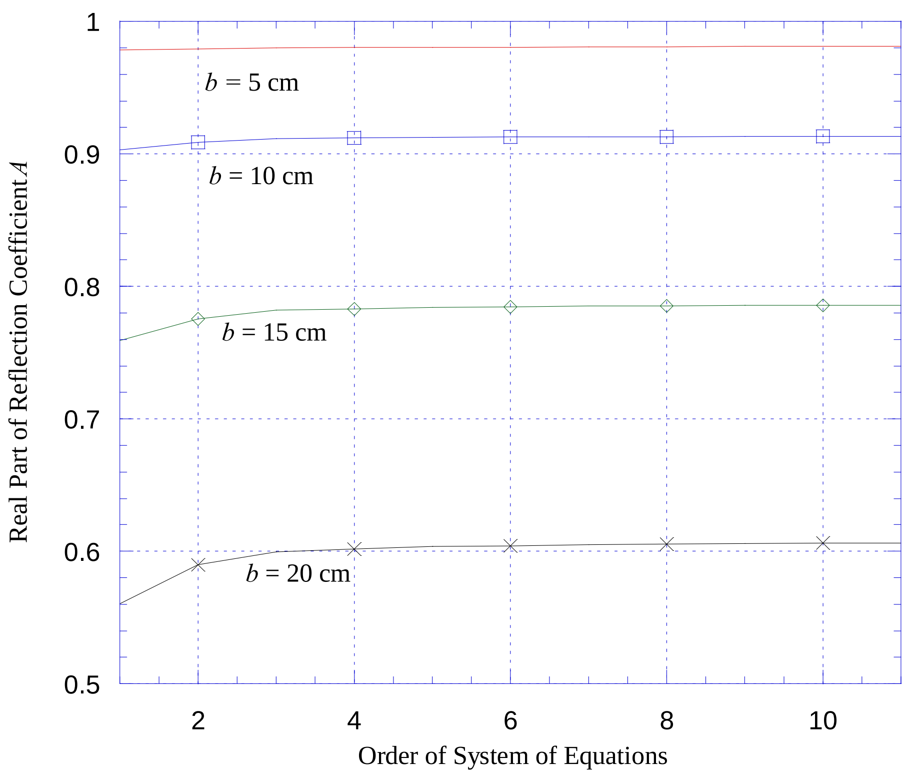

Figure 6.1. The real part of the reflection coefficient *A* as a function of the order of the system of equations. The smallest system is 1x1 and the largest 11x11.

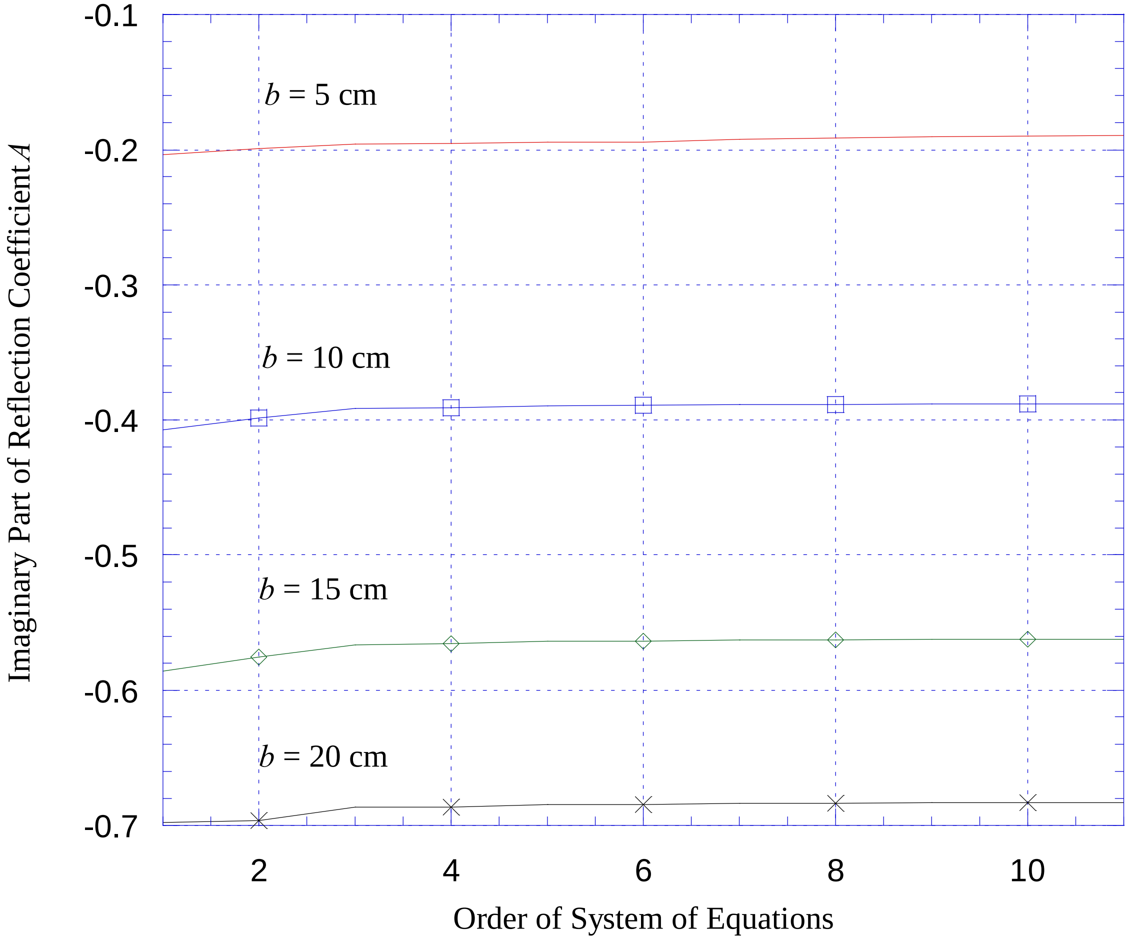

Figure 6.2. The imaginary part of the reflection coefficient A as a function of the order of the system of equations. The smallest system is 1x1 and the largest 11x11.

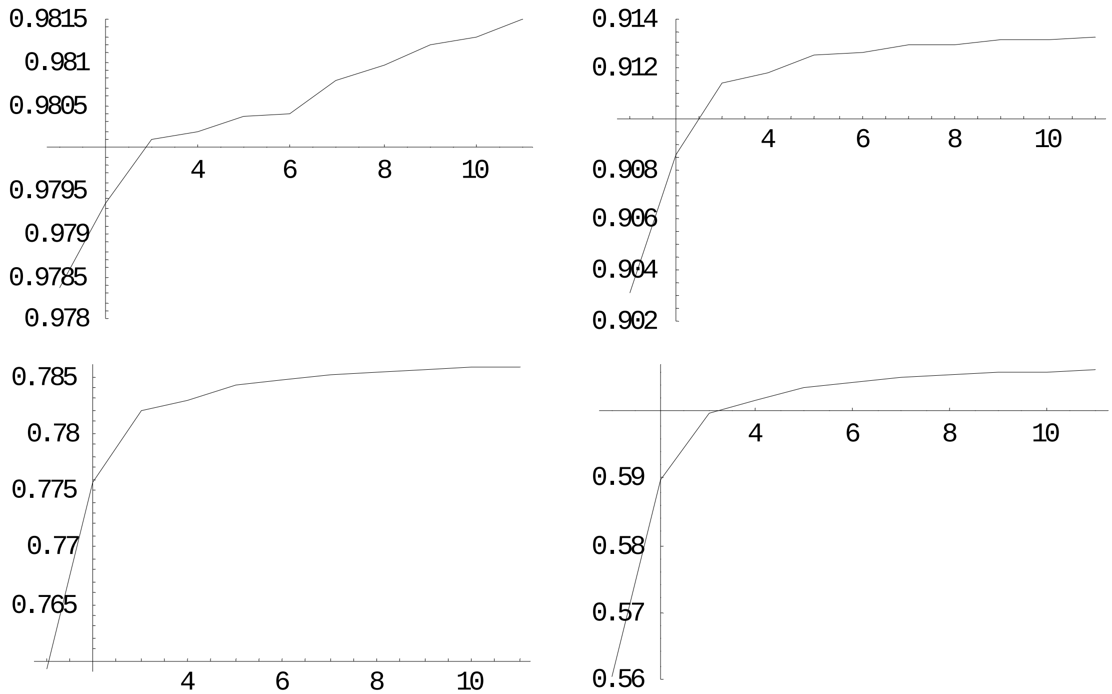

Figure 6.3. Detailed version of Figure 6.1. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

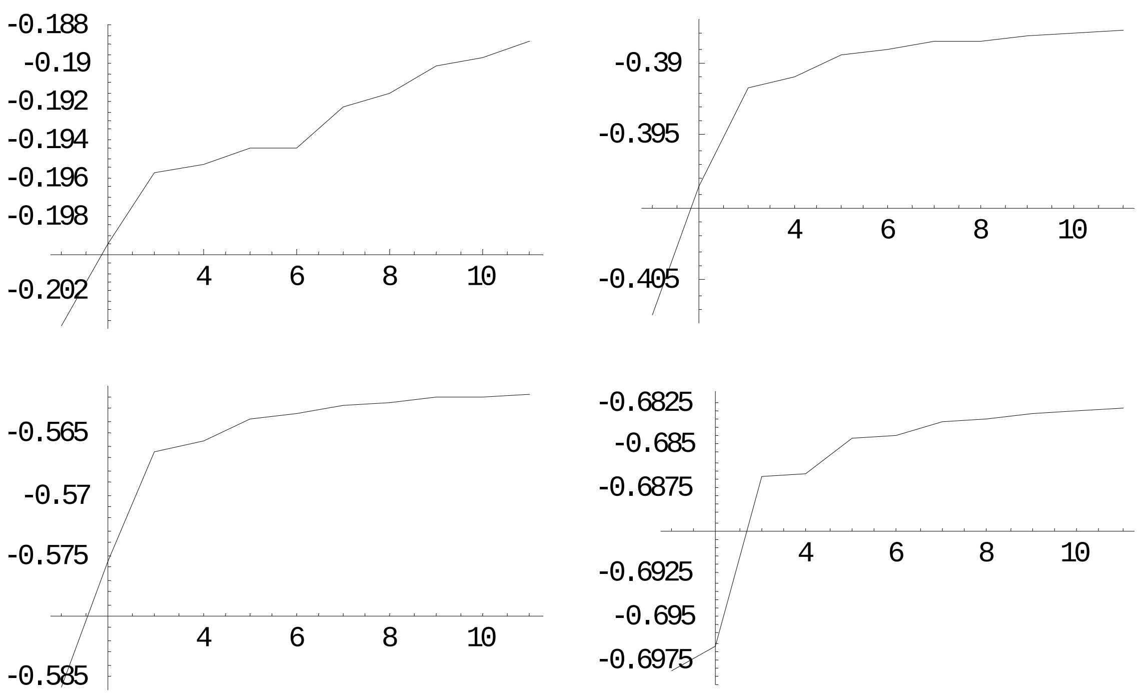

Figure 6.4. Detailed version of Figure 6.2. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

In Figures 6.5 through 6.8, we present the convergence of coefficient B_1. Again, we have convergence to two significant digits.

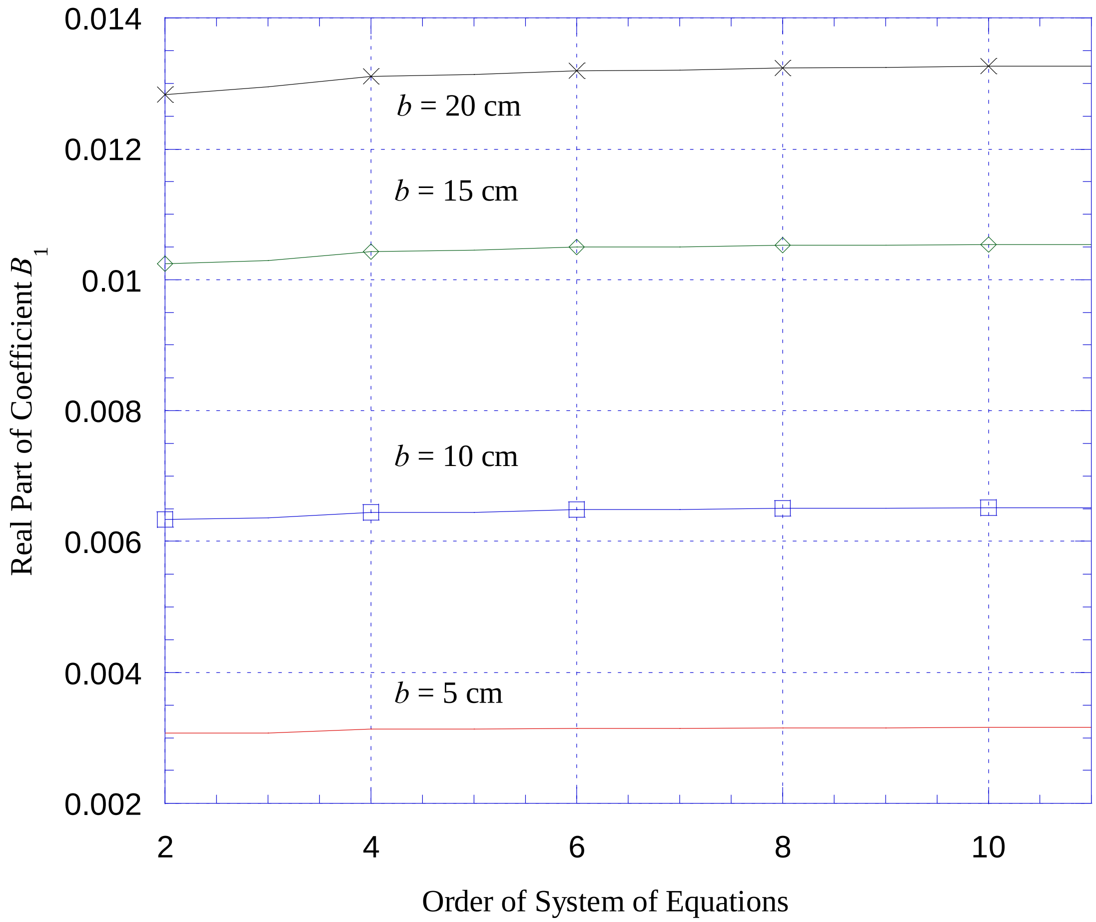

Figure 6.5. The real part of coefficient B_1 as a function of the order of the system of equations.

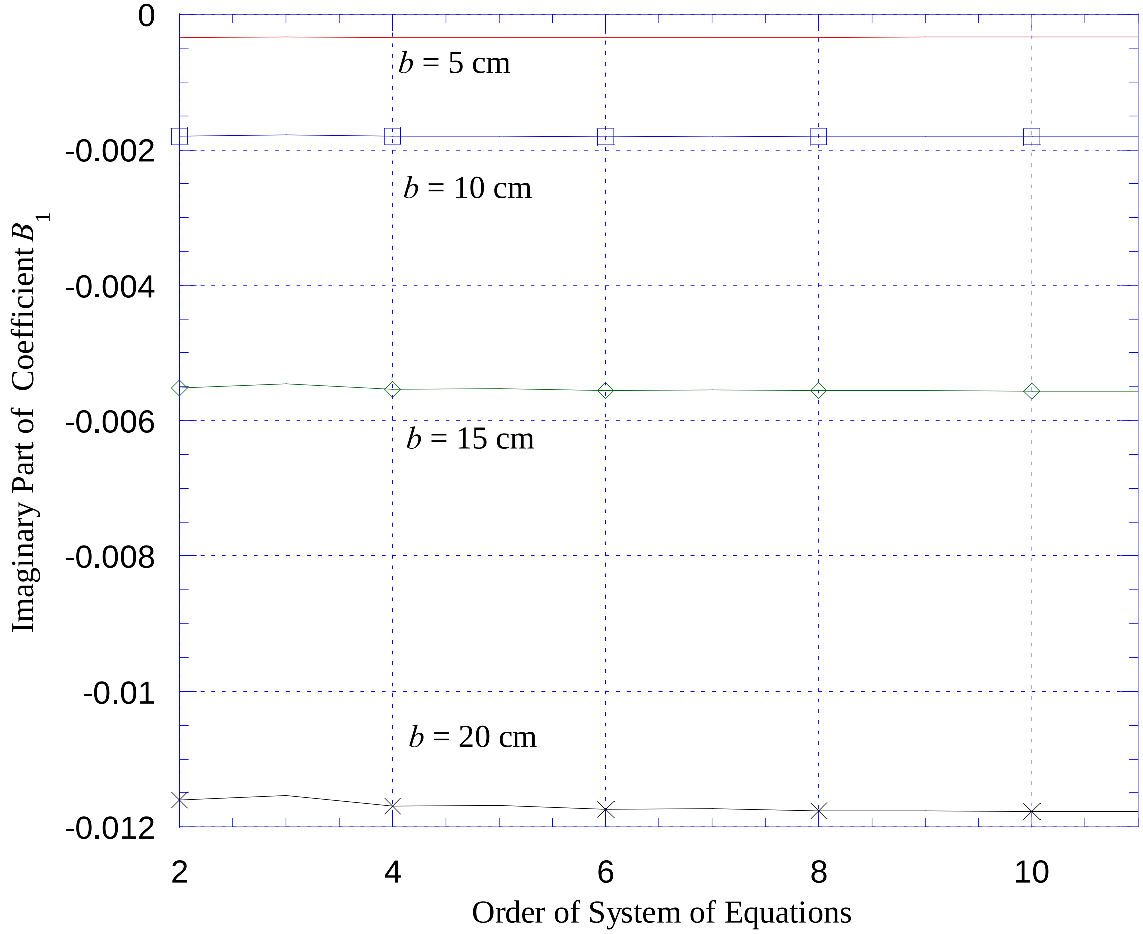

Figure 6.6. The imaginary part of coefficient B_1 as a function of the order of the system of equations. The top curve is for Case 1 (smallest coaxial), the one below it for Case 2, and so on. The smallest system in this case is the 2x2.

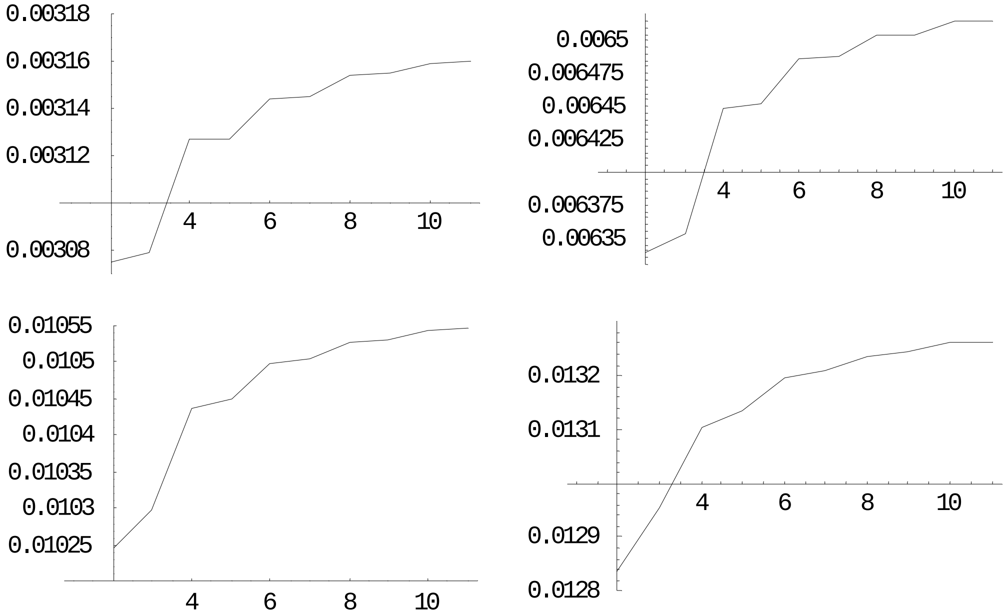

Figure 6.7. Detailed version of Figure 6.5. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

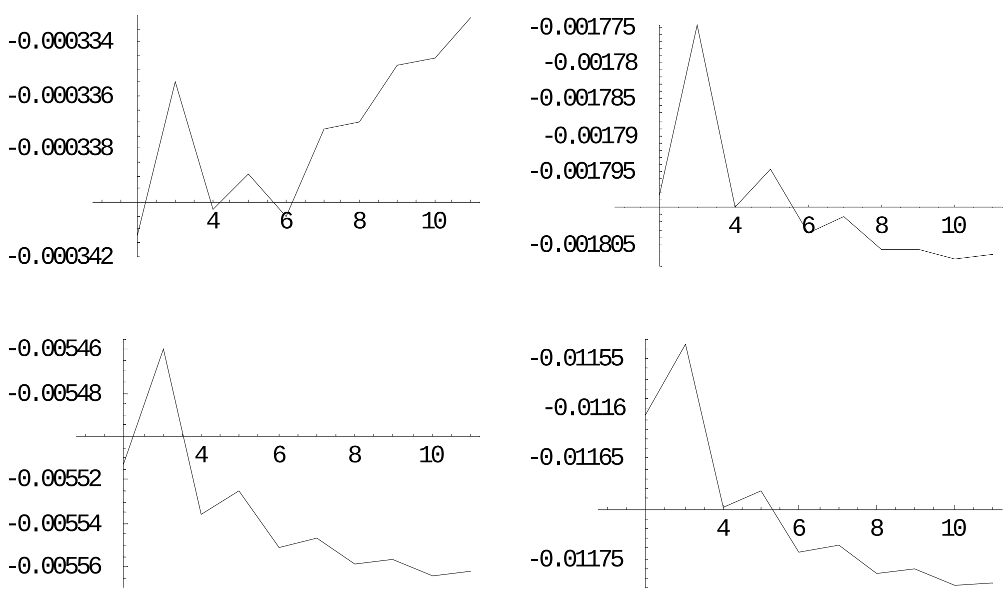

Figure 6.8. Detailed version of Figure 6.6. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

In Figures 6.9 through 6.12, we examine the convergence of coefficient B_2. The results are displayed in Figures 6.9-6.12 and, again, we get convergence to two significant digits.

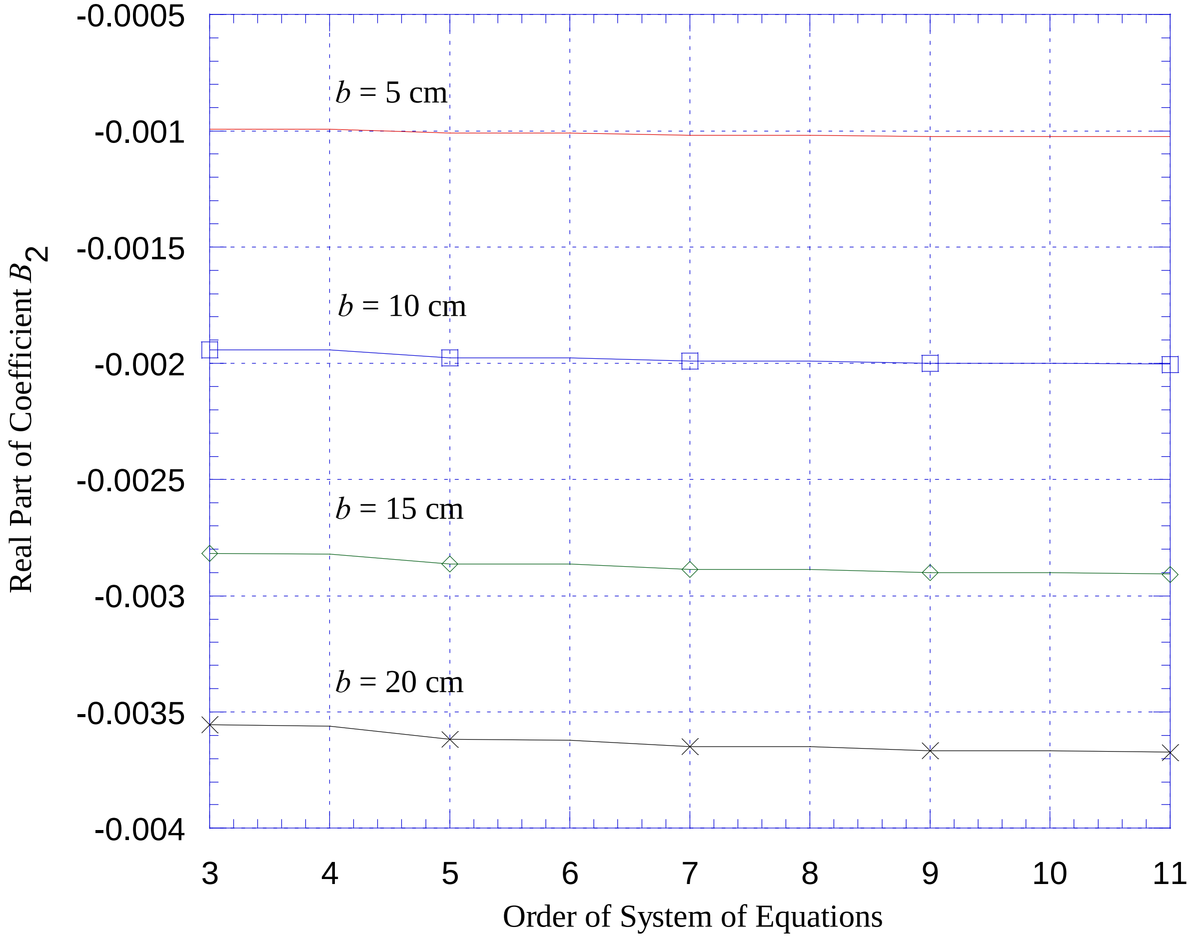

Figure 6.9. The real part of coefficient B_2 as a function of the order of the system of equations. The smallest system is 3x3 and the largest is 11x11.

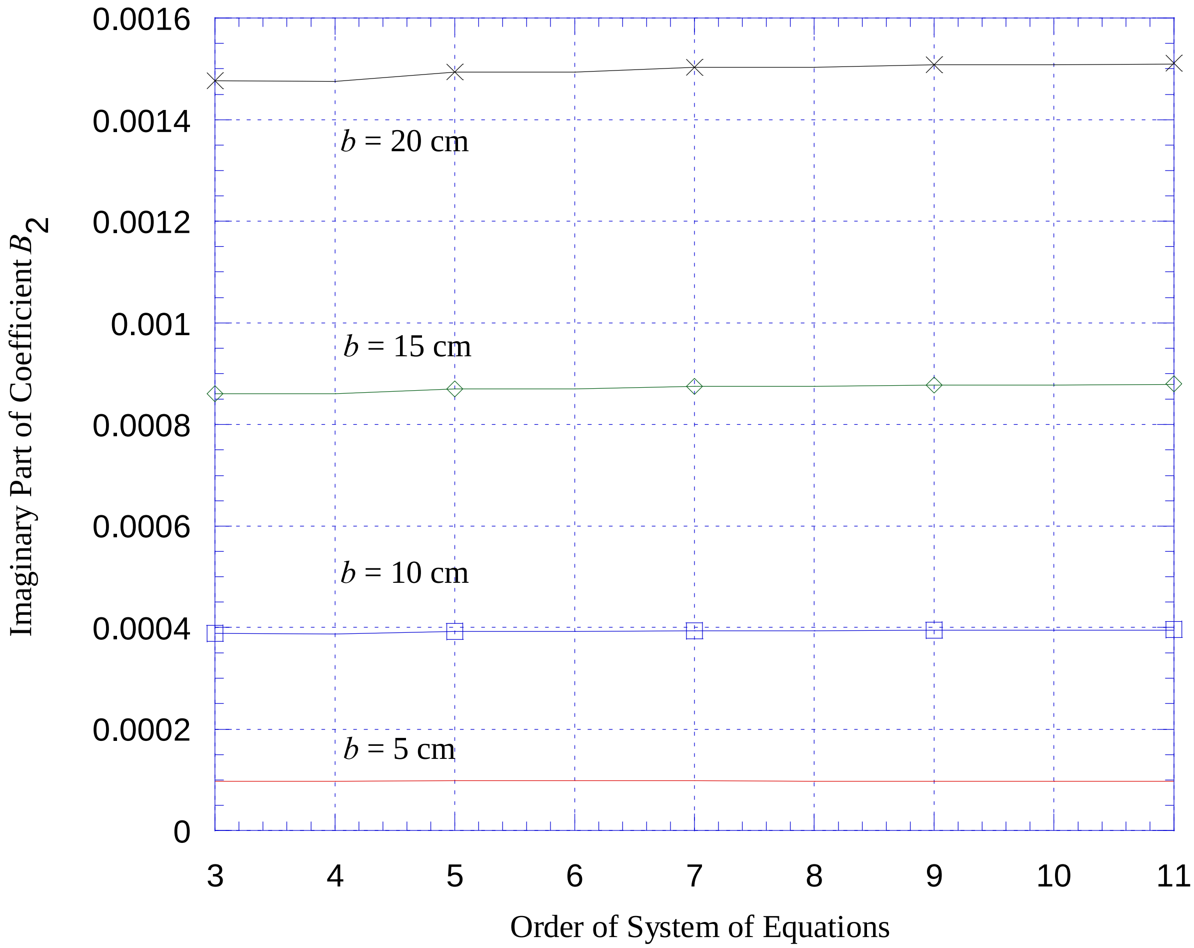

Figure 6.10. The imaginary part of coefficient B_2 as a function of the order of the system of equations. The smallest system is 3x3 and the largest 11x11.

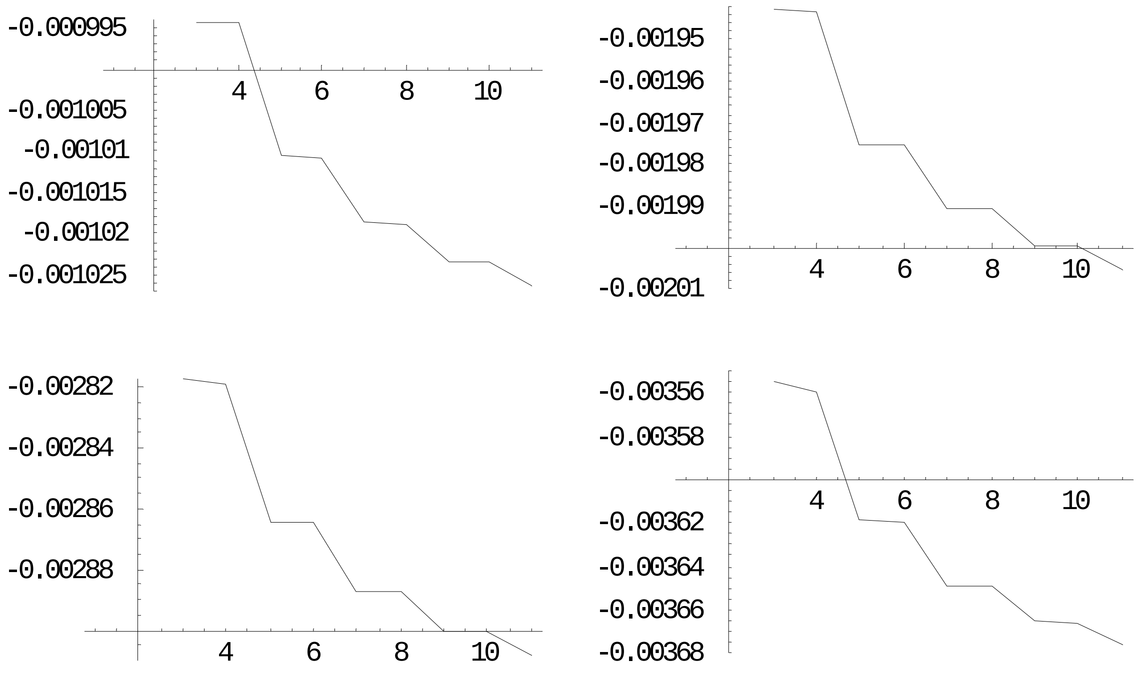

Figure 6.11. Detailed version of Figure 6.9. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

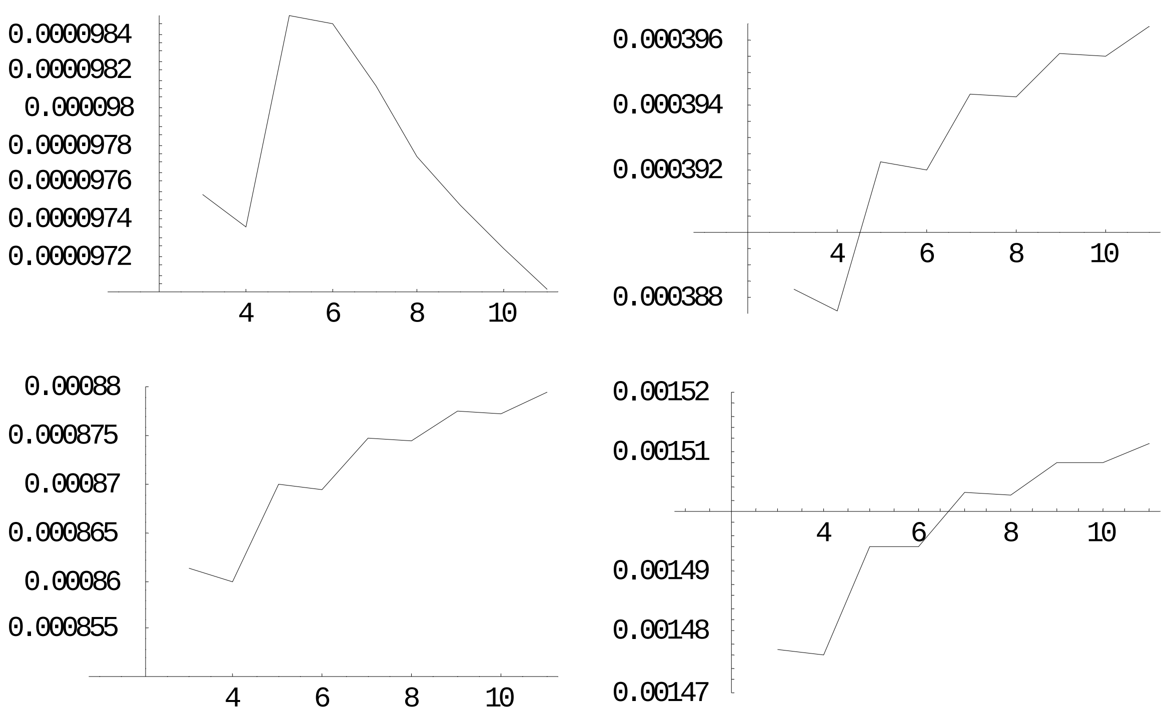

Figure 6.12. Detailed version of Figure 6.10. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

In Figures 6.13 through 6.16, we examine the convergence of coefficient B_3. Convergence appears to be to two significant digits.

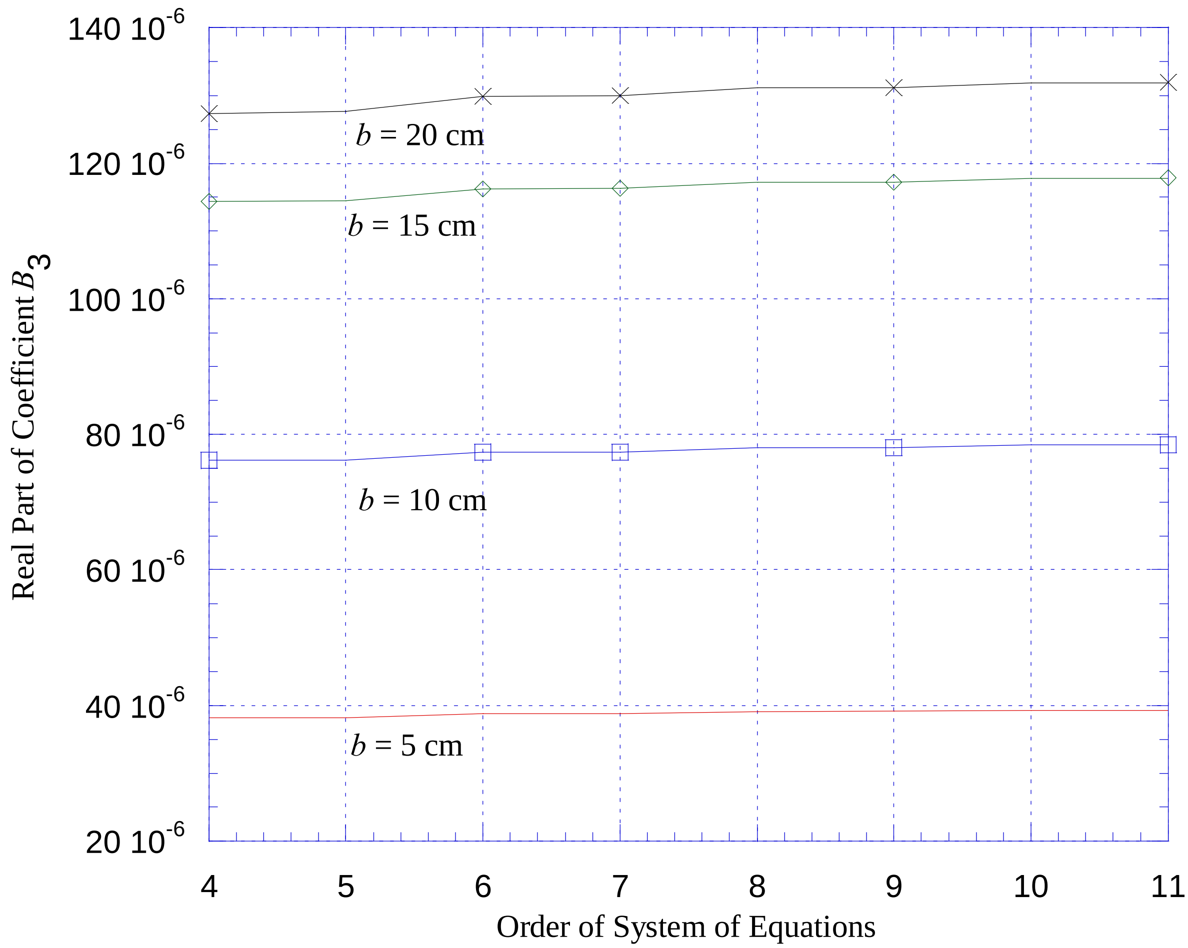

Figure 6.13. The real part of coefficient B_3 as a function of the order of the system of equations. The smallest system is 4x4 and the largest 11x11.

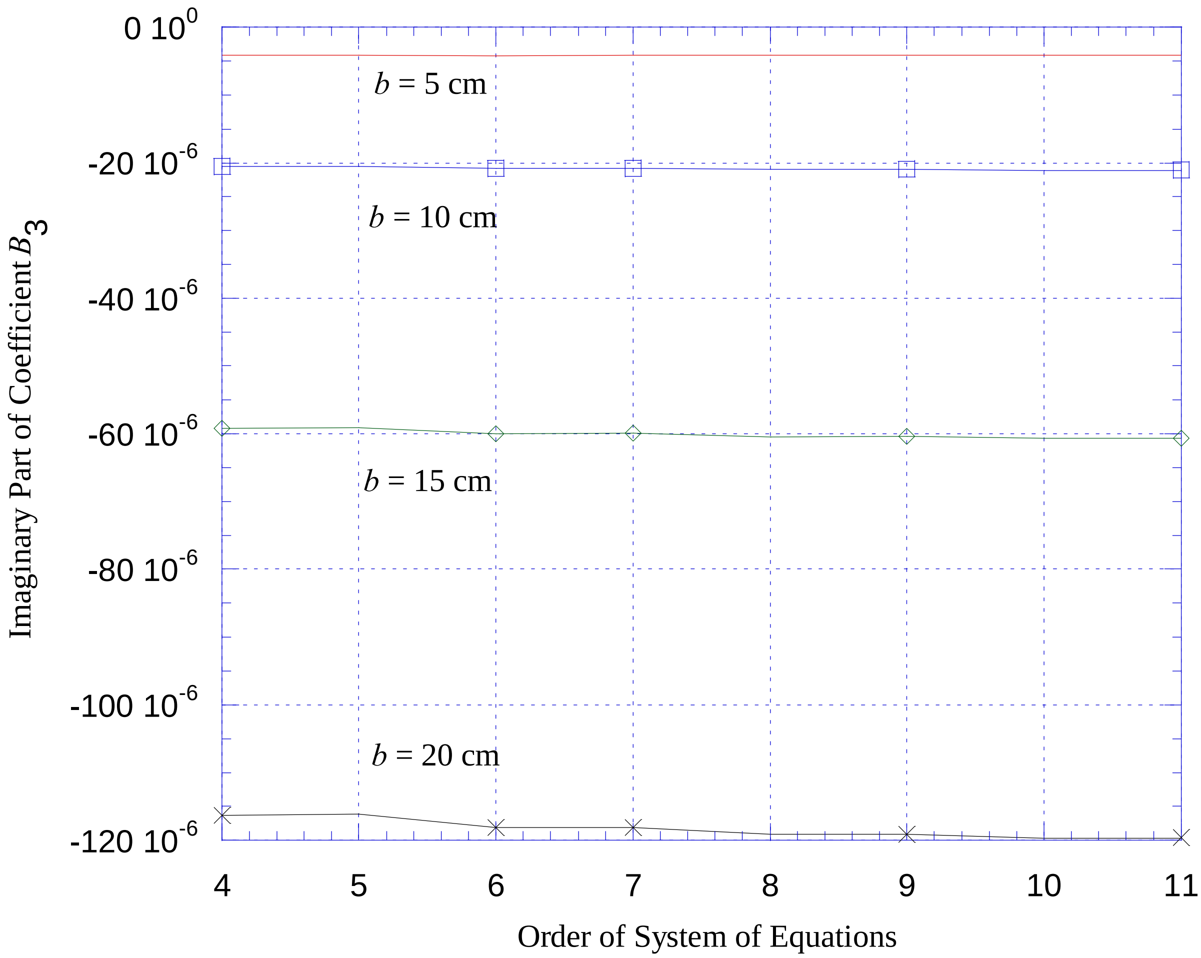

Figure 6.14. The imaginary part of coefficient B_3 as a function of the order of the system of equations. The top curve is for Case 1 (smallest coaxial), the one below it for Case 2, and so on. The smallest system in this case is the 4x4.

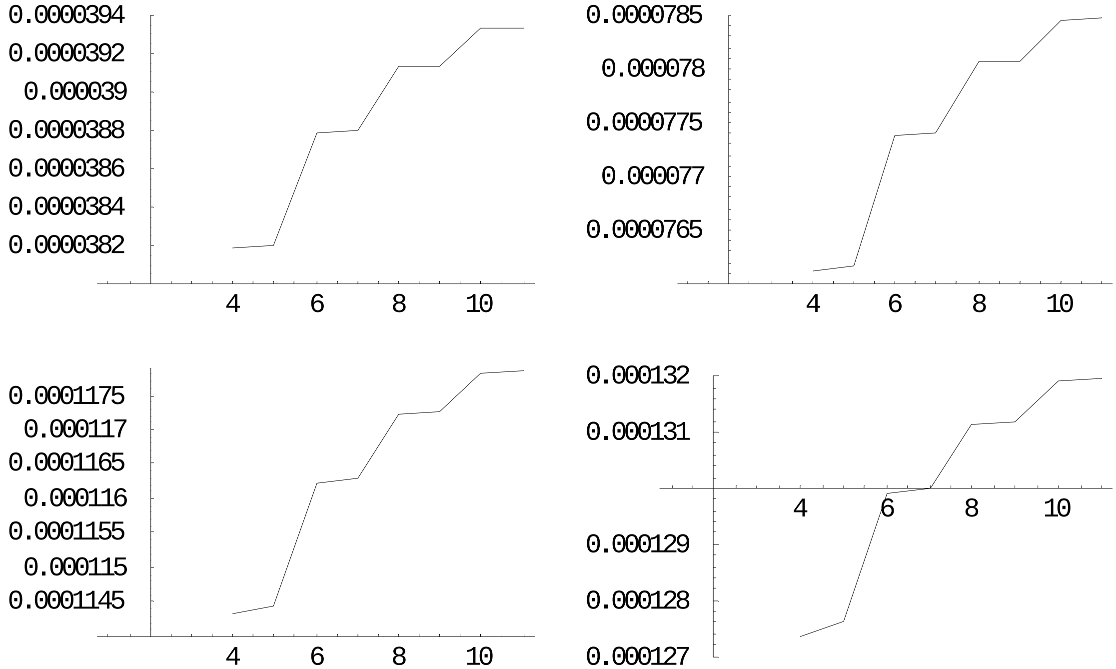

Figure 6.15. Detailed version of Figure 6.13. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

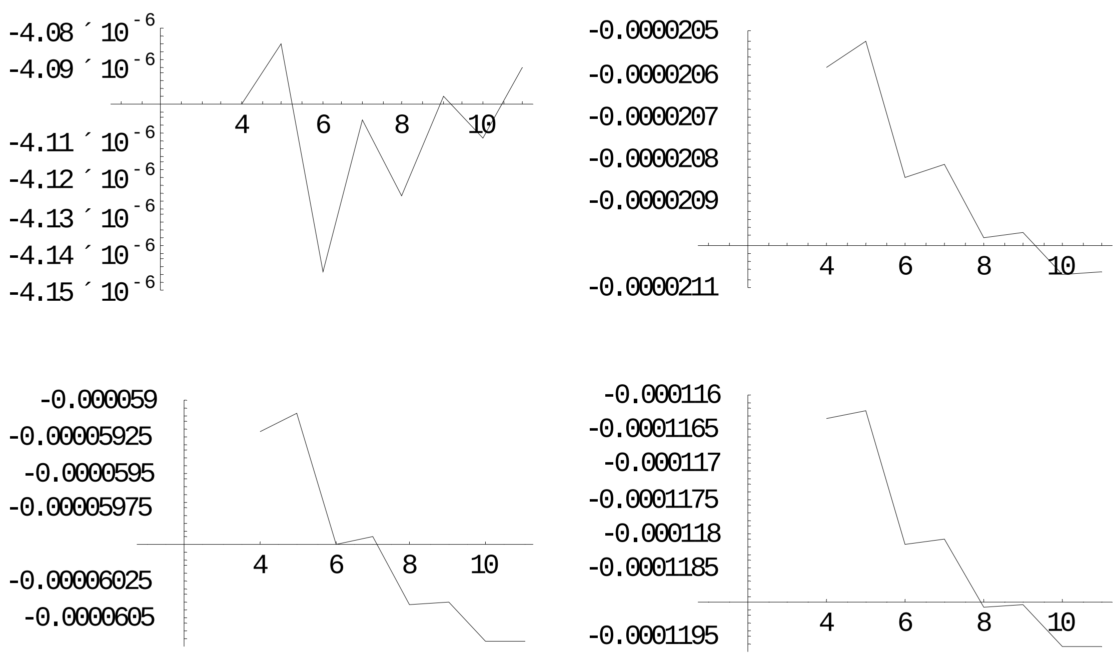

Figure 6.16. Detailed version of Figure 6.14. Case 1 is the top left graph. Case 2 is the top right, Case 3 the bottom left and Case 4 the bottom right.

In Figure 6.17, we have plotted the magnitude of the coefficients B_1 to B_{10}, normalized to the magnitude of the reflection coefficient A. The numbers for the coefficients came out of the 11x11 system. As we saw above, the accuracy of these coefficients does not go beyond the first couple of digits. This graph, however, is useful in displaying their order of magnitude relative to that of A.

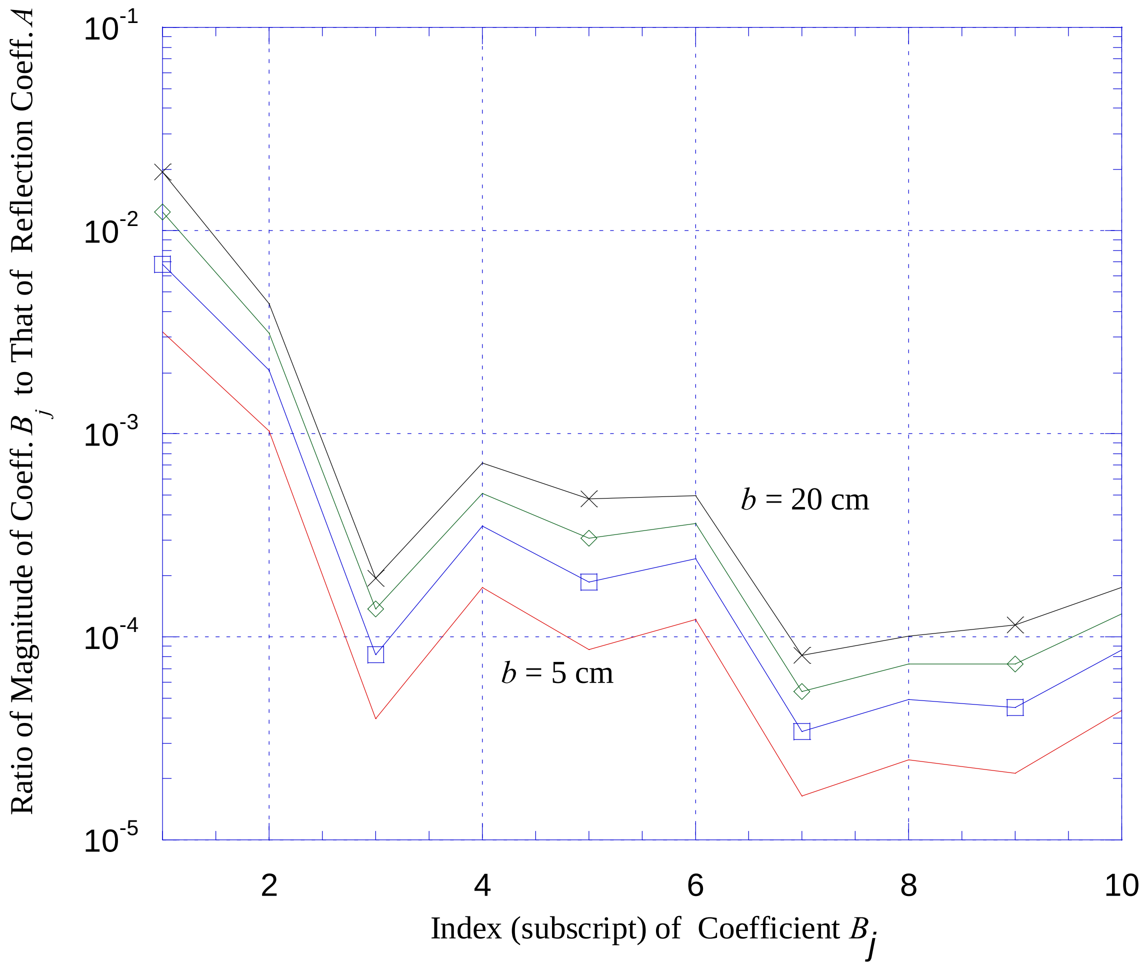

Figure 6.17. Magnitude of coefficients B_1 to B_{10} relative to that of reflection coefficient A. The numbers are those of the 11x11 system.

In Figure 6.18 we plot the real part of A on the horizontal axis and the imaginary on the vertical. The values of the four coax cases are plotted and connected in sequence using straight-line segments. It is clear that, as the dimensions of the coax increase, the real part of A decreases while the absolute value of the imaginary part increases.

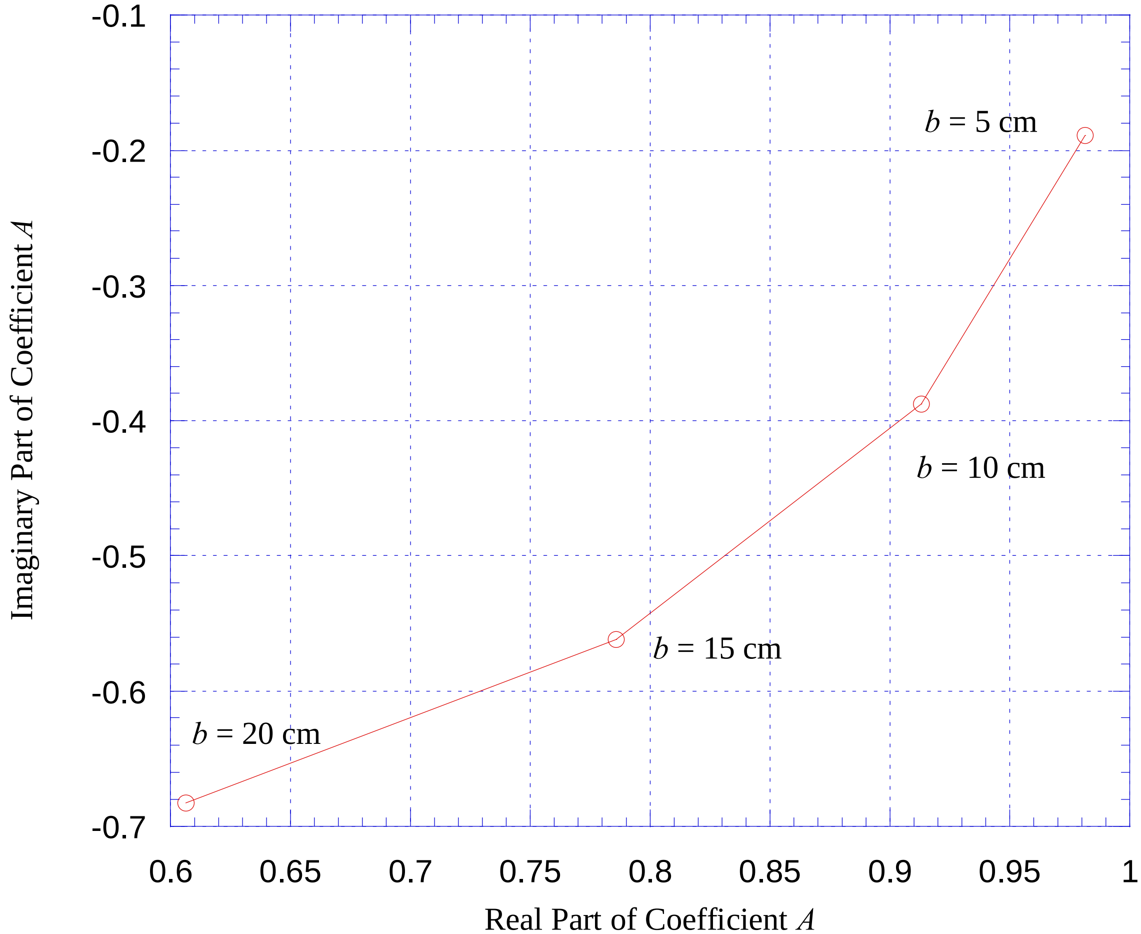

Figure 6.18. The real and imaginary parts of the reflection coefficient A for the four coax sizes. The four computed points are connected using straight-line segments.

From Figures 6.19 and 6.21, for B_1 and B_3, both the real part and the absolute value of the imaginary part increase.

From Figure 6.20, both the absolute value of the real part and the imaginary part of B_2 increase.

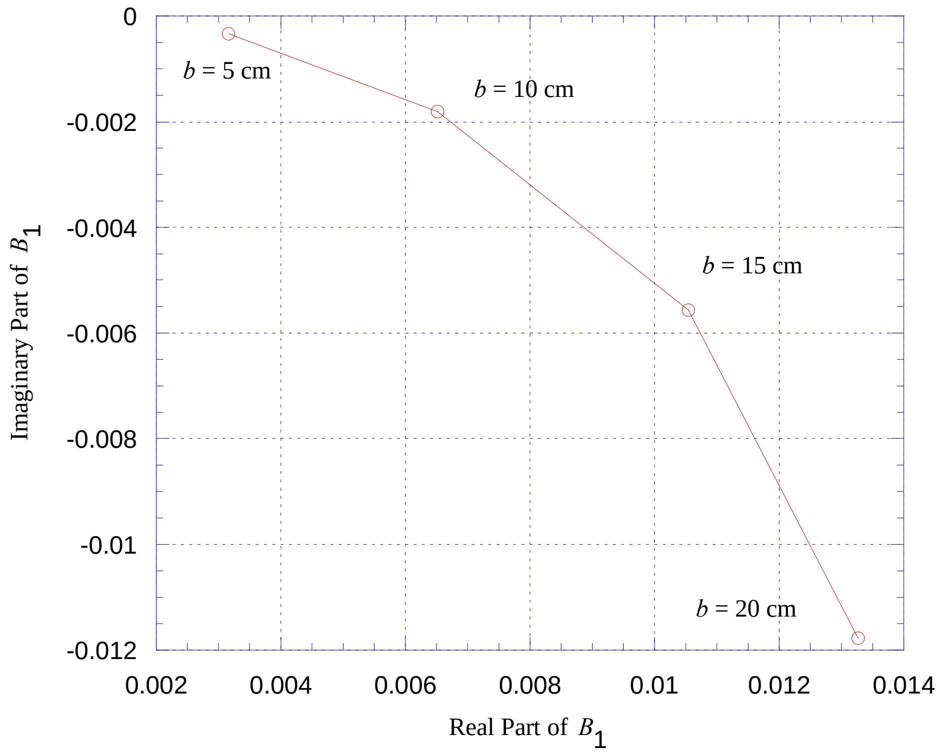

Figure 6.19. The real and imaginary parts of the reflection coefficient B_1 for the four coax sizes. The four computed points are connected using straight-line segments.

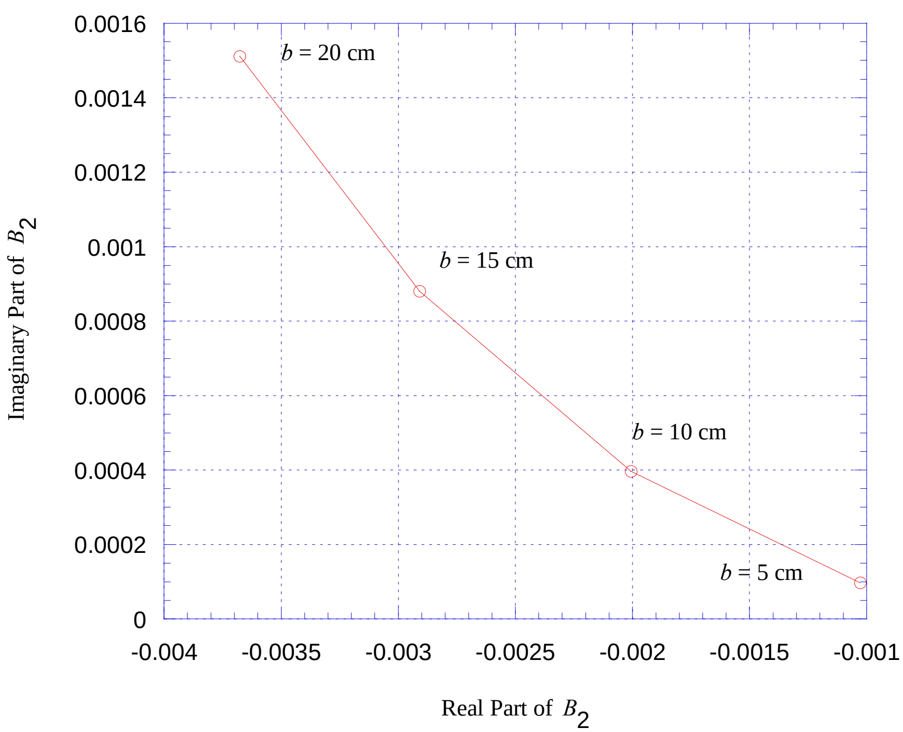

Figure 6.20. The real and imaginary parts of the reflection coefficient B_2 for the four coax sizes. The four computed points are connected using straight-line segments.

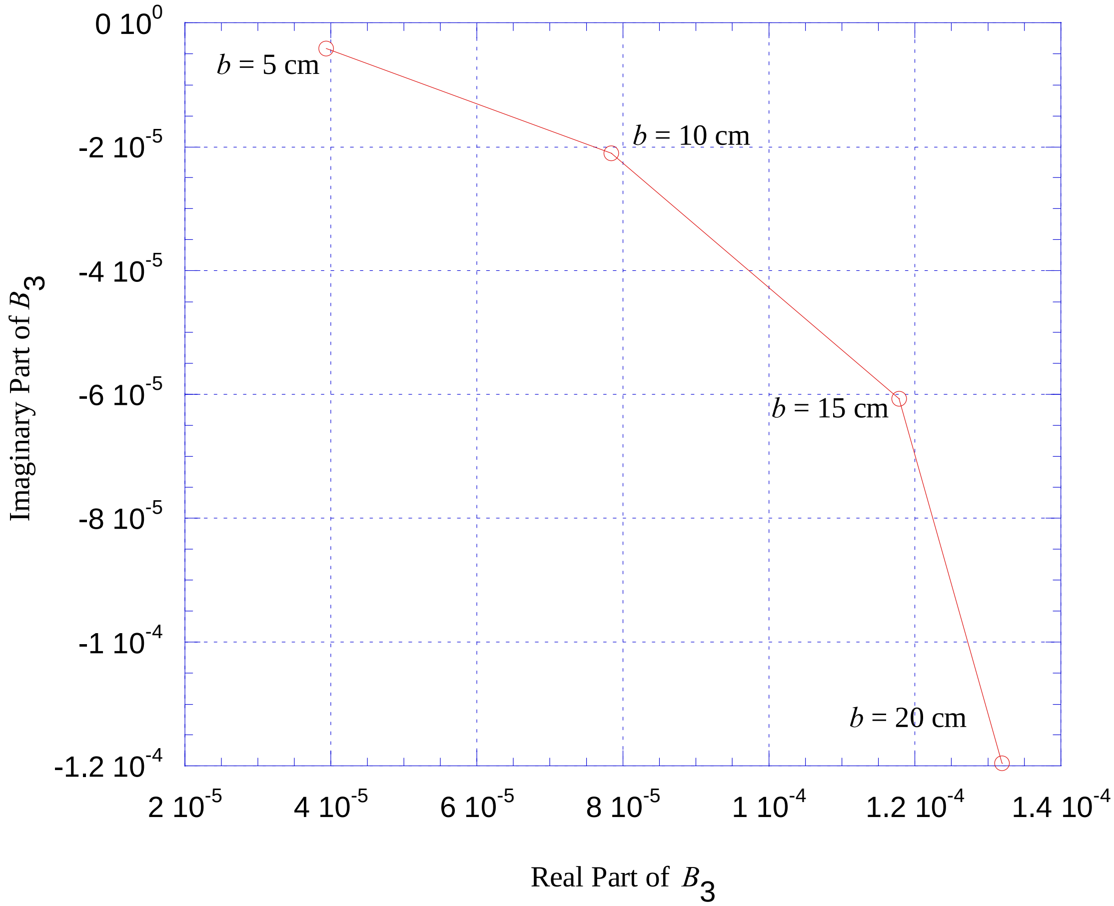

Figure 6.21. The real and imaginary parts of the reflection coefficient B_3 for the four coax sizes. The four computed points are connected using straight-line segments.

7. FAR FIELD

In this section we compute quantities related to the far field. The far field is given by (7.22) and (7.23) of Part 2. For the calculations, we use the coefficients A, B_1, B_2, and B_3. We compute far-field amplitude and phase, far-field intensity (average radiated power density), total radiated average power, directivity, and gain. The definitions we used are the following.

Far-field amplitude and phase:

$$\text{FarFieldAmplitude} = |F(\vartheta)| \tag{7.1}$$

$$\text{FarFieldPhase} = \tan^{-1}\left[\frac{\operatorname{Im}\{F(\vartheta)\}}{\operatorname{Re}\{F(\vartheta)\}}\right] \tag{7.2}$$

where, from (7.22) of Part 2,

$$\begin{aligned} F(\vartheta) &= \frac{1+A}{\ln(\chi)} \frac{J_0(ka\sin\vartheta) - J_0(kb\sin\vartheta)}{\sin\vartheta} \\ &\quad + \frac{2k^2\sin\vartheta}{\pi}\sum_{n=1}^{3} B_n \frac{\sqrt{\nu_{0n}{}^2 - k^2}}{\nu_{0n}{}^2 - k^2\sin^2\vartheta}\left[\frac{J_0(ka\sin\vartheta)}{Y_0(\nu_{0n}a)} - \frac{J_0(kb\sin\vartheta)}{Y_0(\nu_{0n}b)}\right]. \end{aligned} \tag{7.3}$$

Far-field intensity (reference 2, p. 38):

$$\text{FarFieldIntensity} = \frac{|F(\vartheta)|^2}{2Z_0}\,. \tag{7.4}$$

Total radiated average power (reference 2, p. 38):

$$\text{TotalRadiatedAveragePower} = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta \sin\vartheta \frac{|F(\vartheta)|^2}{2Z_0} = \pi Y_0 \int_0^{\pi/2} d\vartheta \sin\vartheta |F(\vartheta)|^2\,. \tag{7.5}$$

Directivity (reference 2, p. 39):

$$\text{Directivity} = \frac{\text{FarFieldIntensity}}{\dfrac{\text{TotalRadiatedAveragePower}}{2\pi}} = 2\pi\frac{\text{FarFieldIntensity}}{\text{TotalRadiatedAveragePower}}\,. \tag{7.6}$$

Average input power:

$$\text{AverageInputPower} = \frac{V^2}{2Z_c} = \frac{1}{100} \tag{7.7}$$

since the characteristic impedance of the line is 50 Ω and the input is one Volt.

Gain (reference 2, p. 58):

$$\text{Gain} = \frac{\text{FarFieldIntensity}}{\frac{\text{AverageInputPower}}{2\pi}} = 2\pi \times \text{FarFieldIntensity} \times 100\,. \tag{7.8}$$

With these definitions, we proceed to display some of the calculations we performed. We begin with the far-field amplitude. The results are shown in Figure 7.1. The polar angle θ is measured from the perpendicular to the infinite plane toward the plane (i.e., it is equal to 90 deg minus elevation). We see that, as the dimensions of the line increase, so is the energy that escapes into the upper-half space. Equivalently, as the operating frequency approaches the cut-off frequency, more and more energy escapes into the upper half space.

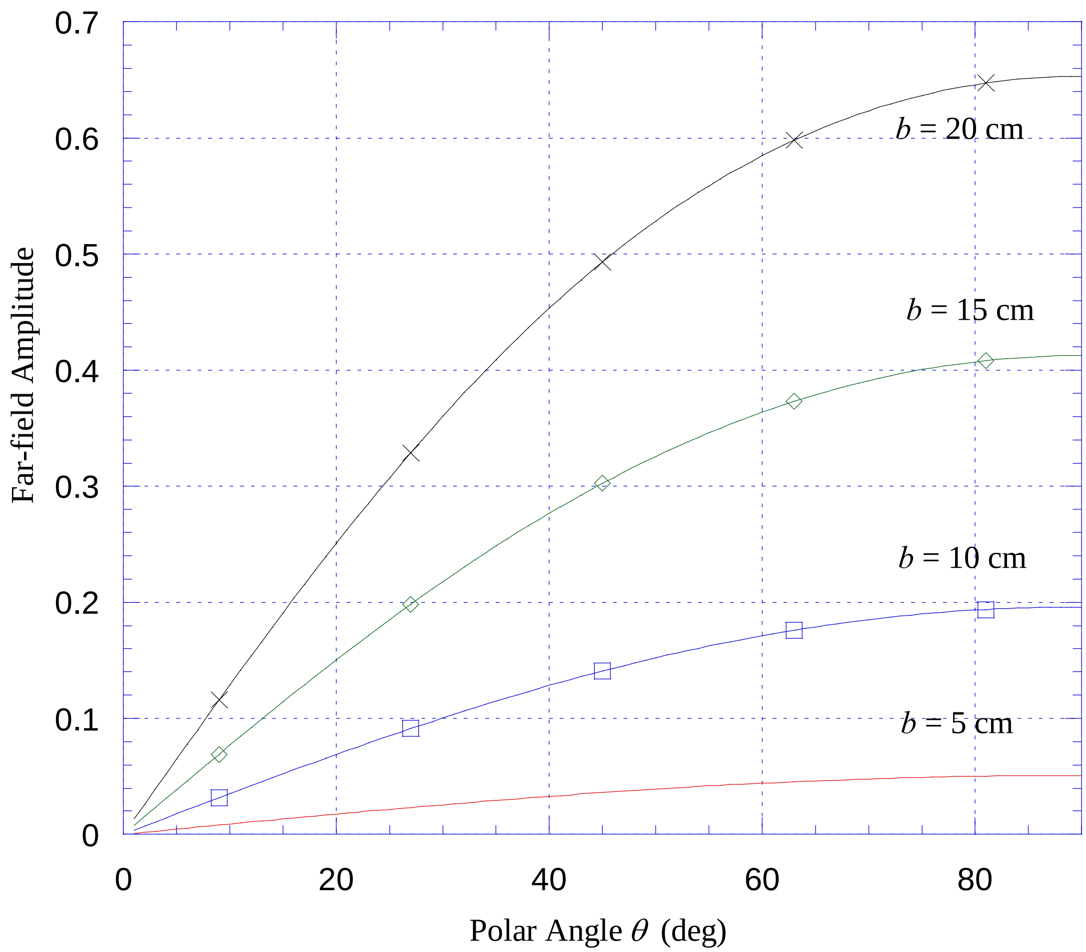

Figure 7.1. Far-field amplitude for the four coaxial cases.

Figure 7.2 shows the phase of the far field. The four different cases are shown separately because of the slow variation of phase. As the size of the coax increases, so does the variation.

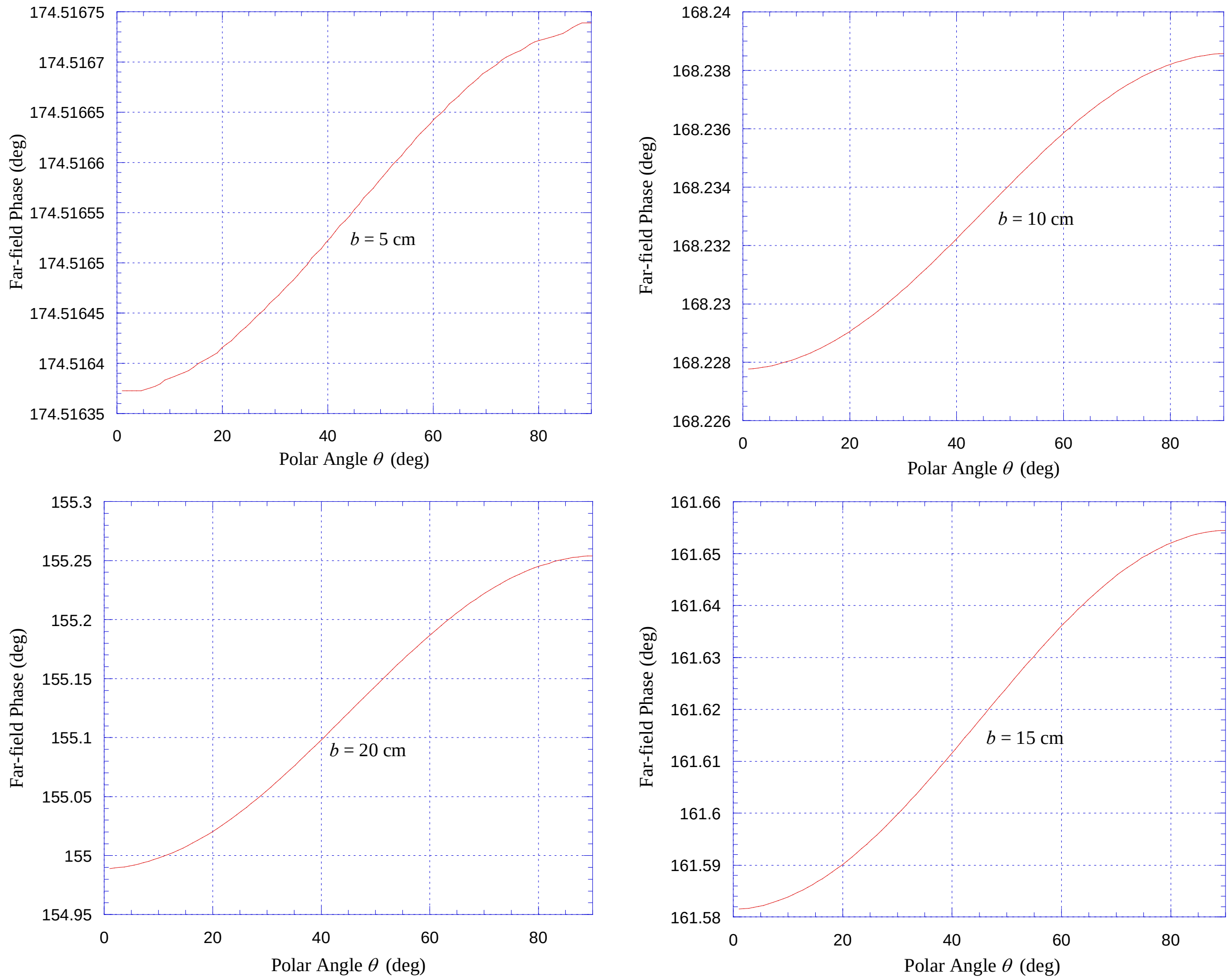

Figure 7.2. Far-field phase for the four coaxial cases.

The results for directivity are shown in Figure 7.3. The variation in the four cases is small but there is a clear tendency for the directivity to rise as the coaxial gets bigger.

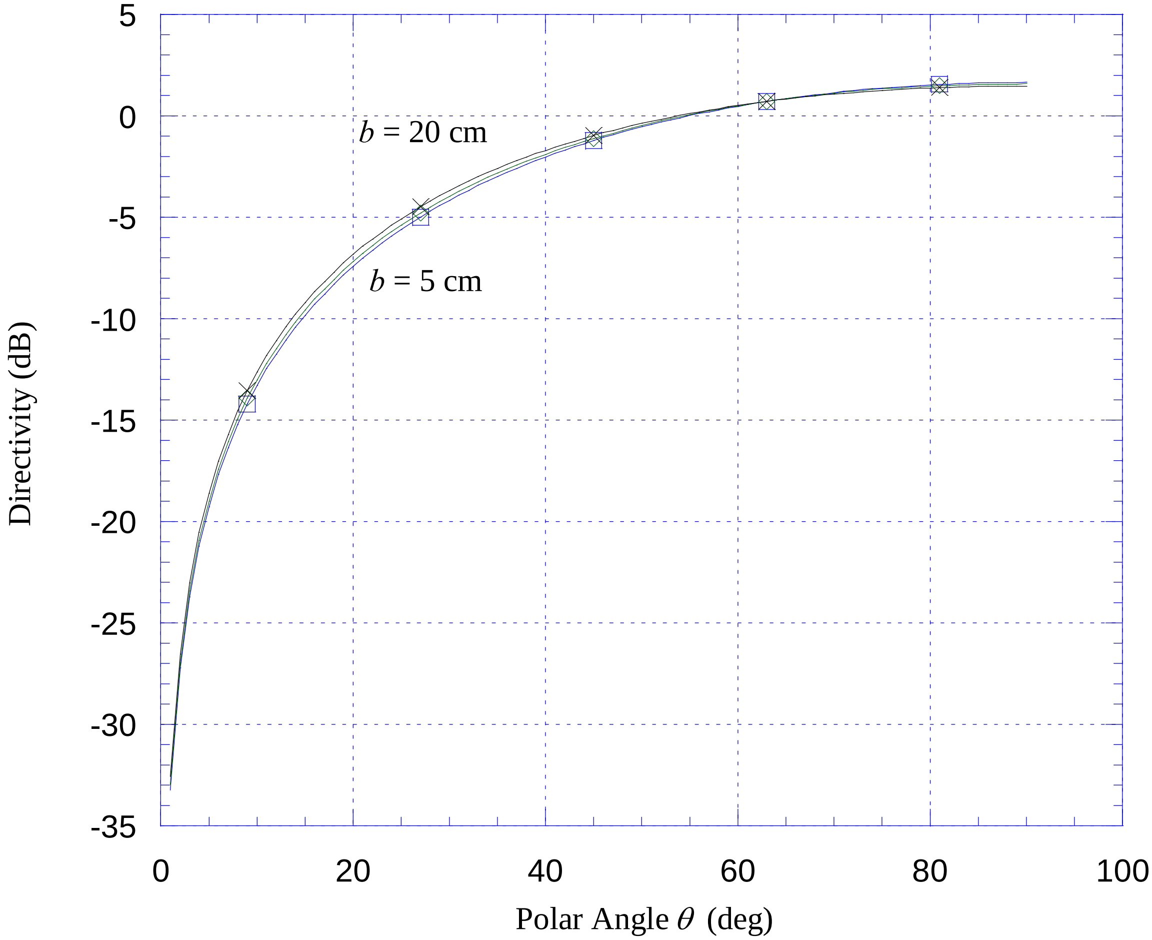

Figure 7.3. Directivity (dB) of four coaxials. Case 1 is the bottom figure while Case 4 is the top one. The rest follow in a clockwise fashion.

The gain results are shown in Figure 7.4. Clearly, the gain improves toward the horizon and with increasing coaxial size.

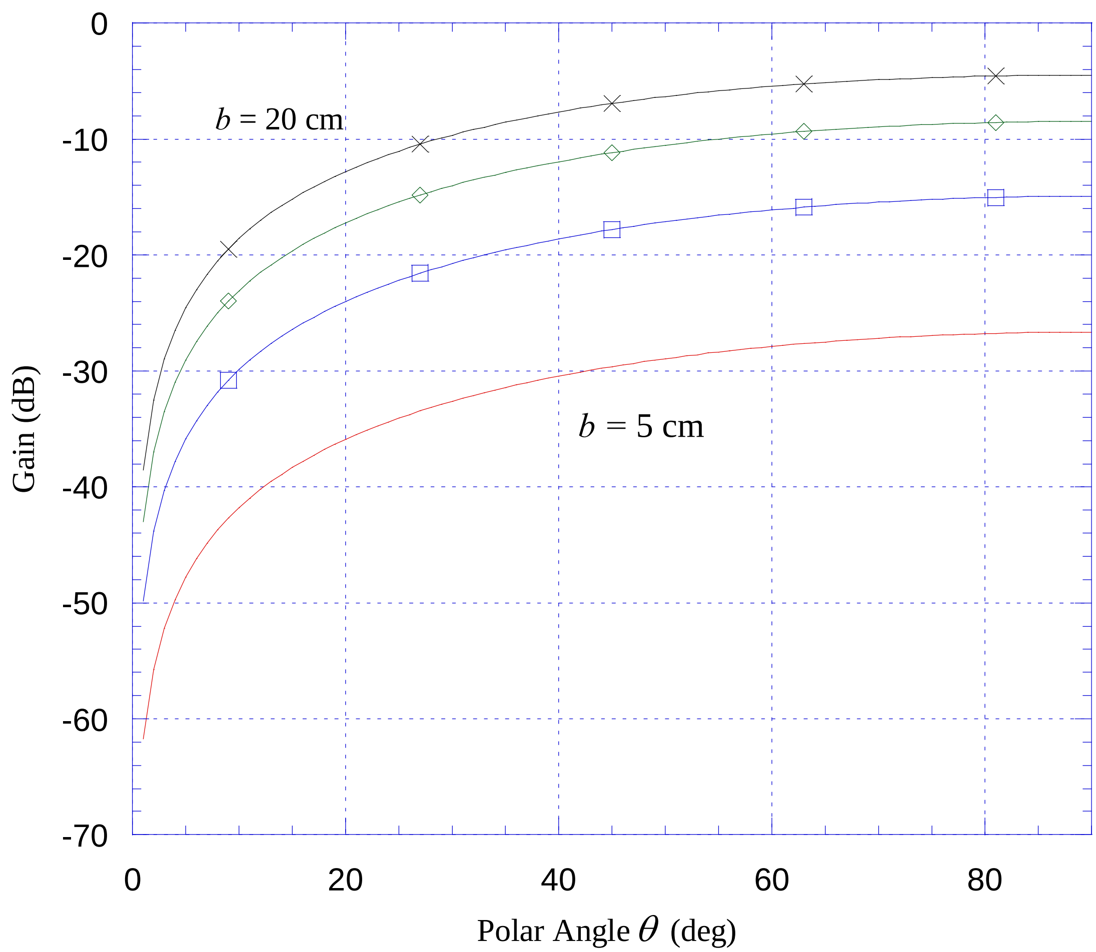

Figure 7.4. Variation of gain with polar angle for the four coaxials.

We have performed additional calculations that appear in the paper that came out of this study (reference 3). There, we compared the coefficient A to that obtained by Bird (reference 4) and found it to be in agreement to two significant digits in all four cases. We also computed gain relative error when using only the 1x1 system rather than the 11x11, and we did the same for the normalized admittance at the opening to the half-space.

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CONCLUSIONS

We have solved the classical problem of radiation by a coaxial line using BIEs. Because of the circular symmetry in the geometry, we can convert the vector integral equations to three scalar equations and we show that, the solution of any of these will determine the unknown current density. We express the latter as an infinite series in the modes of the coaxial line and we determine the unknown coefficients by taking advantage of the orthogonality of the modes. This results in an infinite system of linear equations with the unknowns being the coefficients of the infinite series. In the last part of the report, we have presented numerical examples. The principal asset of this method is that we can provide an engineering solution to the problem by truncating the infinite system to a small system of linear equations. In Part 4, the largest system we used has 11 unknowns (11x11). By contrast, use of other, well known numerical methods, such as finite elements or the method of moments or a hybrid of the two, would require solution of systems with hundreds if not thousands of unknowns. In closing, we wish to mention, that the precise same method can be applied to radiation from a circular or rectangular waveguide or any other guide whose natural modes are known analytically.

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